

文章编号: 1004-4353(2023)03-0209-03

一类 Bézier 型 Szász-Mirakjan-Kantorovich 算子的逼近性质

连博勇

(仰恩大学 数学系, 福建 泉州 362014)

摘要: 研究了一类 Bézier 型 Szász-Mirakjan-Kantorovich 算子. 首先, 给出了该类算子的一阶、二阶中心矩; 然后, 利用 Lipschitz 型不等式得到了该类算子的范数不等式, 并利用二阶连续模和 K 泛函讨论了该算子的逼近性质; 最后, 给出了该类算子对 Lipschitz 函数类的逼近定理.

关键词: Szász-Mirakjan-Kantorovich 算子; 连续模; K 泛函; Lipschitz 函数; 逼近性质

中图分类号: O174.41

文献标志码: A

Approximation property of a class of Szász-Mirakjan-Kantorovich operators of Bézier type

LIAN Boyong

(Department of Mathematics, Yang-En University, Quanzhou 362014, China)

Abstract: A class of Szász-Mirakjan-Kantorovich operators of Bézier type was introduced. Firstly, the first and second order central moment of the operators were given. Then, the norm inequality of the operators was obtained using Lipschitz type inequality. Thirdly, the approximation properties of this type of operators were discussed using second-order modulus of continuity and K-functional. Finally, the approximation theorem of this type of operators for the Lipschitz function class was given.

Keywords: Szász-Mirakjan-Kantorovich operators; modulus of continuity; K-funtional; Lipschitz function; approximation property

0 引言

2021 年, Herzog^[1-2] 研究了一类半指数 Szász-Mirakjan 算子, 其定义为:

$$S_{n,\beta}(\rho, x) = \sum_{k=0}^{\infty} \rho\left(\frac{k}{n}\right) s_{nk}^{\beta}(x), \beta \geq 0. \quad (1)$$

其中: $s_{nk}^{\beta}(x) = e^{-(n+\beta)x} \frac{(n+\beta)^k x^k}{k!}$, $x > 0$. 由式(1)可知: 当 $\beta=0$ 时, 算子 $S_{n,\beta}$ 为经典的 Szász-Mirakjan 算子. 在文献[1-2]中, 作者研究了算子 $S_{n,\beta}$ 的逼近性质, 并讨论了算子 $S_{n,\beta}$ 与其修正形式在收敛速度上的差异. 关于 Szász-Mirakjan 型算子的其他逼近性质, 可以参考文献[3-5]. 在 Herzog 工作的基础上, Agrawal 等^[6] 研究了算子 $S_{n,\beta}$ 的 Kantorovich 变形:

收稿日期: 2023-07-18

作者简介: 连博勇(1982—), 男, 硕士, 教授, 研究方向为函数逼近论.

$$K_{n,\beta}(\rho, x) = n \sum_{k=0}^{\infty} s_{nk}^{\beta}(x) \int_{k/n}^{(k+1)/n} \rho(t) dt. \quad (2)$$

Agrawal 等在文献[6]中不但建立了算子 $K_{n,\beta}$ 和 $S_{n,\beta}$ 的联系以及给出了算子 $K_{n,\beta}$ 的修正形式,而且还讨论了算子 $K_{n,\beta}$ 的 Voronovskaya 型渐近展开式.

由于 Bézier 曲线在计算机辅助设计和计算机图形学中具有重要作用,因此近年来一些学者对其进行了大量研究^[7-9]. 基于上述研究,本文定义了一类 Bézier 型 Szász-Mirakjan-Kantorovich 算子:

$$K_{n,\beta}^{\alpha}(\rho, x) = n \sum_{k=0}^{\infty} Q_{nk}^{\alpha,\beta}(x) \int_{k/n}^{(k+1)/n} \rho(t) dt. \quad (3)$$

其中: $\alpha \geq 1$, $Q_{nk}^{\alpha,\beta}(x) = (J_{n,k}^{\beta}(x))^{\alpha} - (J_{n,k+1}^{\beta}(x))^{\alpha}$, $J_{n,k}^{\beta}(x) = \sum_{j=k}^{\infty} s_{nj}^{\beta}(x)$. 由式(3)可知,当 $\alpha=1$ 时,算子 $K_{n,\beta}^{\alpha}$ 变为 $K_{n,\beta}$. 本文利用二阶连续模和 K 泛函,研究算子 $K_{n,\beta}^{\alpha}$ 的逼近性质,并讨论这类算子对 Lipschitz 函数类的逼近性质.

1 预备知识

首先给出两个定义. 设 $C_B[0, \infty)$ 为有界连续函数空间,任意的函数 $\rho \in C_B[0, \infty)$, 定义 ρ 的范数为 $\|\rho\| = \sup_{x \in [0, \infty)} |\rho(x)|$, Lipschitz 函数类为

$$\text{Lip}_D(\gamma) = \{\rho \mid |\rho(t) - \rho(x)| \leq D |t - x|^{\gamma} \} \quad (0 < \gamma \leq 1, D > 0).$$

引理 1^[6] 设 $x \in [0, \infty)$, $e_i(t) = t^i$, $i = 0, 1, 2$, 则有:

$$\begin{aligned} K_{n,\beta}(e_0, x) &= 1, K_{n,\beta}(e_1, x) = x + \frac{2\beta x + 1}{2n}, \\ K_{n,\beta}(e_2, x) &= x^2 + \frac{3\beta^2 x^2 + 6\beta x(nx + 1) + 6nx + 1}{3n^2}. \end{aligned} \quad (4)$$

引理 2 设 $x \in [0, \infty)$, $\mu_{n,i}(x) = K_{n,\beta}((e_1 - x e_0)^i, x)$, $i = 1, 2$, 则有:

$$\mu_{n,1}(x) = \frac{2\beta x + 1}{2n}, \mu_{n,2}(x) = \frac{3\beta^2 x^2 + 6\beta x + 3nx + 1}{3n^2}. \quad (5)$$

证明 由引理 1 可得:

$$\begin{aligned} \mu_{n,1}(x) &= K_{n,\beta}(e_1 - x, x) = K_{n,\beta}(e_1, x) - x K_{n,\beta}(e_0, x) = \frac{2\beta x + 1}{2n}, \\ \mu_{n,2}(x) &= K_{n,\beta}((e_1 - x)^2, x) = K_{n,\beta}(e_2, x) - 2x K_{n,\beta}(e_1, x) + x^2 K_{n,\beta}(e_0, x) = \\ &= \frac{3\beta^2 x^2 + 6\beta x + 3nx + 1}{3n^2}. \end{aligned}$$

证毕.

引理 3 设 $\rho(x) \in C_B[0, \infty)$, 则有 $\|K_{n,\beta}^{\alpha}(\rho, x)\| \leq \alpha \|\rho\|$.

证明 由 Lipschitz 型不等式可知,对于任意的 $\alpha \geq 1$ 和 $0 \leq y, z \leq 1$ 有 $|y^{\alpha} - z^{\alpha}| \leq \alpha |y - z|$, 因此有 $0 < Q_{nk}^{\alpha,\beta}(x) = (J_{n,k}^{\beta}(x))^{\alpha} - (J_{n,k+1}^{\beta}(x))^{\alpha} \leq \alpha [J_{n,k}^{\beta}(x) - J_{n,k+1}^{\beta}(x)] = \alpha s_{nk}^{\beta}(x)$. 由上式可得:

$$\begin{aligned} |K_{n,\beta}^{\alpha}(\rho, x)| &= \left| n \sum_{k=0}^{\infty} Q_{nk}^{\alpha,\beta}(x) \int_{k/n}^{(k+1)/n} \rho(t) dt \right| \leq n \sum_{k=0}^{\infty} Q_{nk}^{\alpha,\beta}(x) \int_{k/n}^{(k+1)/n} |\rho(t)| dt \leq \alpha \|\rho\| n \sum_{k=0}^{\infty} s_{nk}^{\beta}(x) \cdot \\ &\int_{k/n}^{(k+1)/n} 1 dt = \alpha \|\rho\| K_{n,\beta}(1, x) = \alpha \|\rho\|. \text{ 证毕.} \end{aligned}$$

引理 4^[10] 设 $\bar{\omega}^2[0, \infty) = \{\zeta \mid \zeta', \zeta'' \in C[0, \infty)\}$, 并定义 Peetre-K 泛函为

$$\kappa_2(\rho, v) = \inf_{\zeta \in \bar{\omega}^2[0, \infty)} \{ \|\rho - \zeta\| + v \|\zeta'\| + v^2 \|\zeta''\| \} \quad (v > 0),$$

则存在常数 $\varepsilon > 0$, 使得 $\kappa_2(\rho, v) \leq \varepsilon \bar{\omega}_2(\rho, \sqrt{v})$. 其中 $\bar{\omega}_2$ 是二阶连续模,其定义为:

$$\bar{\omega}_2(\rho, \sqrt{v}) = \sup_{0 < l \leq \sqrt{v}} \sup_{x, x+l, x+2l \in [0, \infty)} |\rho(x+2l) - 2\rho(x+l) + \rho(x)|.$$

2 结果及其证明

定理 1 设 $\rho(x) \in C_B[0, \infty)$, 则存在常数 $\varepsilon > 0$, 使得 $|K_{n,\beta}^a(\rho, x) - \rho(x)| \leq \varepsilon \omega_2\left(\rho, \sqrt{\frac{\alpha \mu_{n,2}(x)}{4}}\right)$.

证明 由泰勒展开式可得 $\zeta(t) = \zeta(x) + \zeta'(x)(t-x) + \int_x^t (t-r)\zeta''(r)dr$, 其中 $\zeta \in \bar{\omega}^2[0, \infty)$, 因此有 $K_{n,\beta}^a(\zeta, x) = \zeta(x) + \zeta'(x)K_{n,\beta}^a(t-x, x) + K_{n,\beta}^a\left(\int_x^t (t-r)\zeta''(r)dr, x\right)$. 于是由 Cauchy-Schwarz 不等式可得 $K_{n,\beta}^a(|t-x|, x) \leq \sqrt{K_{n,\beta}^a(1, x)} \cdot \sqrt{K_{n,\beta}^a((t-x)^2, x)}$. 由于 $K_{n,\beta}^a(1, x) = 1$, $K_{n,\beta}^a((t-x)^2, x) \leq \alpha K_{n,\beta}^a((t-x)^2, x)$, 因此有:

$$\begin{aligned} |K_{n,\beta}^a(\zeta, x) - \zeta(x)| &\leq |\zeta'(x)| |K_{n,\beta}^a(|t-x|, x)| + \left| K_{n,\beta}^a\left(\int_x^t (t-r)\zeta''(r)dr, x\right) \right| \leq \\ &\|\zeta'\| \sqrt{\alpha K_{n,\beta}^a((t-x)^2, x)} + \frac{\|\zeta''\|}{2} \alpha K_{n,\beta}^a((t-x)^2, x) = \|\zeta'\| \sqrt{\alpha \mu_{n,2}(x)} + \frac{\|\zeta''\|}{2} \alpha \mu_{n,2}(x). \end{aligned}$$

由上式可得:

$$\begin{aligned} |K_{n,\beta}^a(\rho, x) - \rho(x)| &\leq |K_{n,\beta}^a(\rho - \zeta, x)| + |\rho(x) - \zeta(x)| + |K_{n,\beta}^a(\zeta, x) - \zeta(x)| \leq \\ &2\|\rho - \zeta\| + \|\zeta'\| \sqrt{\alpha \mu_{n,2}(x)} + \frac{\|\zeta''\|}{2} \alpha \mu_{n,2}(x). \end{aligned}$$

在上述不等式的两边对所有的 $\zeta \in \bar{\omega}^2$ 同时取下确界可得 $|K_{n,\beta}^a(\rho, x) - \rho(x)| \leq 2\kappa_2\left(\rho, \frac{1}{2} \sqrt{\alpha \mu_{n,2}(x)}\right)$,

于是由 $\kappa_2(\rho, v) \leq \varepsilon \omega_2(\rho, \sqrt{v})$ 可知定理 1 成立.

定理 2 设 $\rho(x) \in C_B[0, \infty) \cap \text{Lip}_D(\gamma)$, 则有 $|K_{n,\beta}^a(\rho, x) - \rho(x)| \leq D\alpha(\mu_{n,2}(x))^{\gamma/2}$.

证明 令 $e = \frac{2}{\gamma}$, $f = \frac{2}{2-\gamma}$, 由此显然可得 $\frac{1}{e} + \frac{1}{f} = 1$. 于是利用 Hölder 不等式和式(5) 可得:

$$\begin{aligned} |K_{n,\beta}^a(\rho, x) - \rho(x)| &\leq K_{n,\beta}^a(|\rho(t) - \rho(x)|, x) \leq D \cdot K_{n,\beta}^a(|t-x|^\gamma, x) \leq \\ &D\alpha \cdot K_{n,\beta}^a(|t-x|^\gamma \cdot 1, x) \leq D\alpha \cdot [K_{n,\beta}^a((t-x)^{\gamma e}, x)]^{1/e} \cdot [K_{n,\beta}^a(1^f, x)]^{1/f} = \\ &D\alpha \cdot [K_{n,\beta}^a((t-x)^2, x)]^{1/e} = D\alpha(\mu_{n,2}(x))^{\gamma/2}. \end{aligned}$$

定理 2 证毕.

参考文献:

- [1] HERZOG M. Semi-exponential operators[J]. Symmetry, 2021,13(4):637.
- [2] HERZOG M. Some remarks about exponential and semi-exponential operators[J]. Symmetry, 2021,13(12):2266.
- [3] ZENG X M. On the rate of convergence of the Generalized Szász type operators for functions of bounded variation[J]. J Math Anal Appl, 1998,226:309-325.
- [4] DUMAN O, ÖZARSLAN M A. Szász-Mirakjan type operators providing a better error estimation[J]. Appl Math Lett, 2007,20:1184-1188.
- [5] 连博勇. 修正的 Szász-Mirakjan 算子的逼近性质[J]. 延边大学学报(自然科学版), 2012,38(4):279-281.
- [6] AGRAWAL G, GUPTA V. Approximation properties of semi-exponential Szász-Mirakjan-Kantorovich operators[J]. Filomat, 2023,37(4):1097-1109.
- [7] AGRAWAL P N, ISPIR N, KAJLA A. Approximation properties of Bézie-summation-integral type operators based on Polya-Bernstein function[J]. Appl Math Comput, 2015,259:533-539.
- [8] LIAN B Y, CAI Q B. On the rate of convergence of two generalized Bernstein type operators[J]. Appl Math J Chinese Univ, 2020,35(3):321-331.
- [9] KAJLA A, ÖZGER F, YADAV J. Bézier-Baskakov-Beta type operators[J]. Filomat, 2022,36(19):6735-6750.
- [10] DEVORE R A, LORENTZ G G. Constructive Approximation[M]. Berlin: Springer-Verlag, 1993.