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转移条件下边界中含有谱参数的两类三阶边值问题的研究

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摘要: 在转移条件下利用判断函数和 Rouche 定理研究了如下边界中含有谱参数的两类三阶边值问题：
$\begin{cases}(py'')'+qy=\lambda wy,\\ A_\lambda Y(a)+B_\lambda Y(b)=0,\\ CY(c-)+DY(c+)=0;\end{cases}$ $\begin{cases}(py')''+qy=\lambda wy,\\ A_\lambda Y(a)+B_\lambda Y(b)=0,\\ CY(c-)+DY(c+)=0.\end{cases}$ 研究结果显示，该问题至多有 $m+n+2$ 个特征值.
该结果可为研究奇数阶微分方程边值问题的有限谱提供参考.

关键词: 转移条件；谱参数；判断函数；Rouche 定理
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Two classes of third order boundary value problems with spectral parameters in the boundary under transmission conditions

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Abstract: Using the iterative construction of the characteristic function and Rouche's theorem under transmission conditions, the following two classes of third order boundary value problems with spectral parameters are studied:

$$\begin{cases}(py'')'+qy=\lambda wy,\\ A_\lambda Y(a)+B_\lambda Y(b)=0,\\ CY(c-)+DY(c+)=0;\end{cases}\quad \begin{cases}(py')''+qy=\lambda wy,\\ A_\lambda Y(a)+B_\lambda Y(b)=0,\\ CY(c-)+DY(c+)=0.\end{cases}$$

The results of the study showed that problems had at most $m+n+2$ eigenvalues. The results can provide a reference for the finite spectrum of boundary value problems of odd order differential equations.

Keywords: transmission conditions; spectral parameters; characteristic function; Rouche's theorem

0　引言

近年来，学者们对 Sturm-Liouville 有限谱问题进行了较多研究[1-7]，但对于奇数阶具有有限谱的微分方程边值问题研究得较少. 2013 年，Ao 等[8]讨论了一类边界条件带有谱参数且具有转移条件的正则 Sturm-Liouville 问题的特征值数量. 2017 年，Ao[9]研究了如下两类三阶边值问题：

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$$\begin{cases}(py'')'+qy=\lambda wy,\\AY(a)+BY(b)=0;\end{cases}\quad\begin{cases}(py')''+qy=\lambda wy,\\AY(a)+BY(b)=0.\end{cases}$$

研究显示,对于每一个正整数 m,上述问题至多有 $2m+1$ 个特征值.受上述文献启发,本文对如下转移条件下边界中含有谱参数的两类三阶边值问题进行研究:

$$\begin{cases}(py'')'+qy=\lambda wy,\\A_\lambda Y(a)+B_\lambda Y(b)=0,\\CY(c-)+DY(c+)=0;\end{cases}\tag{1}$$

$$\begin{cases}(py')''+qy=\lambda wy,\\A_\lambda Y(a)+B_\lambda Y(b)=0,\\CY(c-)+DY(c+)=0.\end{cases}\tag{2}$$

其中:$y=y(t)$;$t\in J=(a,c)\cup(c,b)$;$-\infty<a<b<+\infty$;$A_\lambda=\begin{pmatrix}\lambda\alpha'_1-\alpha_1 & -\lambda\alpha'_2+\alpha_2 & \lambda\alpha'_3+\alpha_3\\1&1&1\\0&0&0\end{pmatrix}$;

$B_\lambda=\begin{pmatrix}0&0&0\\0&0&0\\\lambda\beta'_1+\beta_1 & -\lambda\beta'_2-\beta_2 & \lambda\beta'_3-\beta_3\end{pmatrix}$;$\alpha_i,\alpha'_i,\beta_i,\beta'_i\in R(i=1,2,3)$,且 $\alpha_i,\alpha'_i,\beta_i,\beta'_i\in R$ 满足 $\begin{vmatrix}\alpha_1&\alpha_2&\alpha_3\\\alpha'_1&\alpha'_2&\alpha'_3\\1&1&1\end{vmatrix}\neq0$,$\begin{vmatrix}\beta_1&\beta_2&\beta_3\\\beta'_1&\beta'_2&\beta'_3\\1&1&1\end{vmatrix}\neq0$;$C,D\in M_3R$,$|C|=\rho>0$,$|D|=\theta>0$;$\lambda$ 为谱参数;系统 $(py'')'+qy=\lambda wy$ 和 $(py')''+qy=\lambda wy$ 中的系数 p、q 和 w 满足如下基本条件:

$$r=1/p,\ q,w\in L(J,C),\tag{3}$$

其中 $L(J,C)$ 为 Lebesgue 可积的复值函数在 J 上构成的集合.

1 预备知识

令 $u_1=y$,$u_2=y'$,$u_3=py''$,则与方程 $(py'')'+qy=\lambda wy$ 等价的系统可表示为:

$$u'_1=u_2,\ u'_2=ru_3,\ u'_3=(\lambda w-q)u_1.\tag{4}$$

式(4)的矩阵形式为 $\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}'=\begin{pmatrix}0&1&0\\0&0&r\\\lambda w-q&0&0\end{pmatrix}\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}$.

定义 1 设 $y=y(t)$ 为问题的解.若 $y\equiv0$,$u_2=y'\equiv0$,$u_3=py''\equiv0$,则称 y 为问题的平凡解,反之称为非平凡解.

引理 1 设 $\Phi(x,\lambda)=[\phi_{ij}(x,\lambda)]$ 是系统(4)满足初始条件 $\Phi(a,\lambda)=I$ 的基解矩阵,则 $\lambda\in C$ 是式(1)的特征值当且仅当

$$\Delta(\lambda)=\det[A_\lambda+B_\lambda\Phi(b,\lambda)].\tag{5}$$

特别地

$$\Delta(\lambda)=\sum_{i=1,j=1}^{3}h_{ij}(\lambda)\phi_{ij}(b,\lambda)=H(\lambda)\Phi(b,\lambda),\tag{6}$$

其中:

$$H(\lambda):=\begin{pmatrix}h_{11}(\lambda)&h_{12}(\lambda)&h_{13}(\lambda)\\h_{21}(\lambda)&h_{22}(\lambda)&h_{23}(\lambda)\\h_{31}(\lambda)&h_{32}(\lambda)&h_{33}(\lambda)\end{pmatrix},$$

$h_{11}(\lambda)=-(\lambda\beta'_1+\beta_1)(\lambda\alpha'_2-\alpha_2+\lambda\alpha'_3+\alpha_3)$, $h_{12}(\lambda)=(\lambda\beta'_1+\beta_1)(\lambda\alpha'_3+\alpha_3-\lambda\alpha'_1+\alpha_1)$,

$h_{13}(\lambda)=-(\lambda\beta'_1+\beta_1)(\lambda\alpha'_2+\alpha_2-\lambda\alpha'_1+\alpha_1)$, $h_{21}(\lambda)=(\lambda\beta'_2+\beta_2)(\lambda\alpha'_3+\alpha_3+\lambda\alpha'_2-\alpha_2)$,

$h_{22}(\lambda)=(\lambda\beta'_2+\beta_2)(\lambda\alpha'_1-\alpha_1-\lambda\alpha'_3-\alpha_3)$, $h_{23}(\lambda)=-(\lambda\beta'_2+\beta_2)(\lambda\alpha'_2-\alpha_2+\lambda\alpha'_1-\alpha_1)$,

$h_{31}(\lambda)=-(\lambda\beta'_3-\beta_3)(\lambda\alpha'_2-\alpha_2+\lambda\alpha'_3+\alpha_3)$, $h_{32}(\lambda)=(\lambda\beta'_3-\beta_3)(\lambda\alpha'_3+\alpha_3-\lambda\alpha'_1+\alpha_1)$,

$h_{33}(\lambda)=(\lambda\beta'_3-\beta_3)(\lambda\alpha'_1-\alpha_1-\lambda\alpha'_2-\alpha_2)$.

证明　首先证明必要性.设 $\Delta(\lambda)=0$,则由平凡解和非平凡解的定义可知 $[A_\lambda+B_\lambda\Phi(b,\lambda)]C=0$ 有一个非平凡向量解.为验证 $\lambda\in C$ 是式(1)的特征值,解初值问题 $Y'=\begin{pmatrix}0&1&0\\0&0&1/p\\\lambda w-q&0&0\end{pmatrix}Y$, $Y(a)=C$ 即可.解该问题得 $Y(b)=\Phi(b,\lambda)Y(a)$, $[A_\lambda+B_\lambda\Phi(b,\lambda)]Y(a)=0$.由此可得,$Y$ 的第 1 个分量 y 是问题(1)的特征函数,且 λ 是问题(1)的一个特征值.

下证充分性.若 $\lambda\in C$ 是问题(1)的特征值,且 Y 的第 1 个分量 y 为 λ 所对应的特征函数,则 $Y=\begin{pmatrix}y\\y'\\py''\end{pmatrix}$ 满足 $Y(b)=\Phi(b,\lambda)Y(a)$.由此可知,$[A_\lambda+B_\lambda\Phi(b,\lambda)]Y(a)=0$.若 $Y(a)=0$,则 y 是平凡解,这与 y 是特征函数矛盾,故有 $\Delta(\lambda)=\det[A_\lambda+B_\lambda\Phi(b,\lambda)]=0$.再对式(5)进行行列式计算即可得式(6),由此引理 1 得证.

2　三阶含谱参数问题的有限谱

首先假设 $J=(a,c)\cup(c,b)$ 存在如下划分:

$$\begin{aligned}&a=a_0<a_1<a_2<\cdots<a_{2m}<a_{2m+1}=c,\\&c=b_0<b_1<b_2<\cdots<b_{2n}<b_{2n+1}=b.\end{aligned}\tag{7}$$

由式(7)可知,对于问题(1),当 $r(t)=\dfrac{1}{p(t)}=0$ 时有:

$$\begin{aligned}&\int_{a_{2k}}^{a_{2k+1}}w(t)\mathrm{d}t\neq0,\ \int_{a_{2k}}^{a_{2k+1}}w(t)t\,\mathrm{d}t\neq0,\ k=0,1,\cdots,m,\ t\in(a_{2k},a_{2k+1}),\\&\int_{b_{2i}}^{b_{2i+1}}w(t)\mathrm{d}t\neq0,\ \int_{b_{2i}}^{b_{2i+1}}w(t)t\,\mathrm{d}t\neq0,\ i=0,1,\cdots,n,\ t\in(b_{2i},b_{2i+1}).\end{aligned}\tag{8}$$

当 $q(t)=w(t)=0$ 时有:

$$\begin{aligned}&\int_{a_{2k+1}}^{a_{2k+2}}r(t)\mathrm{d}t\neq0,\ \int_{a_{2k+1}}^{a_{2k+2}}r(t)t\,\mathrm{d}t\neq0,\ k=0,1,\cdots,m-1,\ t\in(a_{2k+1},a_{2k+2}),\\&\int_{b_{2i+1}}^{b_{2i+2}}r(t)\mathrm{d}t\neq0,\ \int_{b_{2i+1}}^{b_{2i+2}}r(t)t\,\mathrm{d}t\neq0,\ i=0,1,\cdots,n-1,\ t\in(b_{2i+1},b_{2i+2}).\end{aligned}\tag{9}$$

为便于后续计算,在式(7)—(9)的基础上本文补充如下相关条件:

$$\begin{aligned}&r_k=\int_{a_{2k+1}}^{a_{2k+2}}r,\ \hat{r}_k=\int_{a_{2k+1}}^{a_{2k+2}}r(t)t\,\mathrm{d}t,\ k=0,1,\cdots,m-1;\\&q_k=\int_{a_{2k}}^{a_{2k+1}}q,\ \hat{q}_k=\int_{a_{2k}}^{a_{2k+1}}q(t)t\,\mathrm{d}t,\ k=0,1,\cdots,m;\\&w_k=\int_{a_{2k}}^{a_{2k+1}}w,\ \hat{w}_k=\int_{a_{2k}}^{a_{2k+1}}w(t)t\,\mathrm{d}t,\ k=0,1,\cdots,m;\\&\tilde{r}_i=\int_{b_{2i+1}}^{b_{2i+2}}r,\ \breve{r}_i=\int_{b_{2i+1}}^{b_{2i+2}}r(t)t\,\mathrm{d}t,\ i=0,1,\cdots,n-1;\\&\tilde{q}_i=\int_{b_{2i}}^{b_{2i+1}}q,\ \breve{q}_i=\int_{b_{2i}}^{b_{2i+1}}q(t)t\,\mathrm{d}t,\ i=0,1,\cdots,n;\end{aligned}\tag{10}$$

$\widetilde{w}_i=\int_{b_{2i}}^{b_{2i+1}}w$, $\breve{w}_i=\int_{b_{2i}}^{b_{2i+1}}w(t)t\,\mathrm{d}t$, $i=0,1,\cdots,n$.

引理 2 令 $\Phi(t,\lambda)=[\phi_{ij}(t,\lambda)]$ 是系统(4)满足初始条件 $\Phi(a,\lambda)=I$ 的基解矩阵,则有

$$\Phi(a_1,\lambda)=\begin{pmatrix}1 & a_1-a_0 & 0\\ 0 & 1 & 0\\ \lambda w_0-q_0 & (\lambda w_0-q_0)(a_1-a_0) & 1\end{pmatrix},\tag{11}$$

$$\Phi(a_3,\lambda)=\begin{pmatrix}\phi_{11}(a_3,\lambda) & \phi_{12}(a_3,\lambda) & r_0(a_3-a_1)\\ r_0(\lambda w_0-q_0) & 1+r_0(\lambda w_0-q_0)(a_1-a_0) & r_0\\ \phi_{31}(a_3,\lambda) & \phi_{32}(a_3,\lambda) & (\lambda w_1-q_1)[r_0(a_3-a_1)]+1\end{pmatrix}.\tag{12}$$

其中:$\phi_{11}(a_3,\lambda)=(a_3-a_1)r_0(\lambda w_0-q_0)+1$, $\phi_{12}(a_3,\lambda)=(a_3-a_1)[r_0(\lambda w_0-q_0)(a_1-a_0)+1]+(a_1-a_0)$, $\phi_{31}(a_3,\lambda)=[(a_3-a_1)r_0(\lambda w_0-q_0)+1](\lambda w_1-q_1)+(\lambda w_0-q_0)$, $\phi_{32}(a_3,\lambda)=(\lambda w_1-q_1)(a_3-a_1)r_0+1$. 当 $1\leqslant i\leqslant m$ 时有:

$$\Phi(a_{2i+1},\lambda)=\begin{pmatrix}1 & 1 & r_{i-1}\\ 0 & 1 & r_{i-1}\\ \lambda w_i-q_i & \lambda w_i-q_i & r_{i-1}(\lambda w_i-q_i)\end{pmatrix}\Phi(a_{2i-1},\lambda).\tag{13}$$

证明 由式(4)可知,在 r 恒等于零的子区间上 u_2 是常数,在 q 和 w 恒等于零的子区间上 u_3 是常数. 于是根据 $\Phi(a_1,\lambda)$ 和 $\Phi(a_3,\lambda)$ 以及式(4)即可递推出引理 2 中 $\Phi(a_{2i+1},\lambda)$ 的结构,引理 2 得证.

引理 3 令 $\Psi(t,\lambda)=[\psi_{ij}(t,\lambda)]$ 是系统(4)满足初始条件 $\Psi(c,\lambda)=I$ 的基解矩阵,则有

$$\Psi(b_1,\lambda)=\begin{pmatrix}1 & b_1-b_0 & 0\\ 0 & 1 & 0\\ \lambda\widetilde{w}_0-\tilde{q}_0 & (\lambda\widetilde{w}_0-\tilde{q}_0)(b_1-b_0) & 1\end{pmatrix},\tag{14}$$

$$\Psi(b_3,\lambda)=\begin{pmatrix}\psi_{11}(b_3,\lambda) & \psi_{12}(b_3,\lambda) & \tilde{r}_0(b_3-b_1)\\ \tilde{r}_0(\lambda\widetilde{w}_0-\tilde{q}_0) & 1+\tilde{r}_0(\lambda\widetilde{w}_0-\tilde{q}_0)(b_1-b_0) & \tilde{r}_0\\ \psi_{31}(b_3,\lambda) & \psi_{32}(b_3,\lambda) & (\lambda\widetilde{w}_1-\tilde{q}_1)[\tilde{r}_0(b_3-b_1)]+1\end{pmatrix},\tag{15}$$

其中:$\psi_{11}(b_3,\lambda)=(b_3-b_1)\tilde{r}_0(\lambda\widetilde{w}_0-\tilde{q}_0)+1$, $\psi_{12}(b_3,\lambda)=(b_3-b_1)[\tilde{r}_0(\lambda\widetilde{w}_0-\tilde{q}_0)(b_1-b_0)+1]+(b_1-b_0)$, $\psi_{31}(b_3,\lambda)=[(b_3-b_1)\tilde{r}_0(\lambda\widetilde{w}_0-\tilde{q}_0)+1](\lambda\widetilde{w}_1-\tilde{q}_1)+(\lambda\widetilde{w}_0-\tilde{q}_0)$, $\psi_{32}(b_3,\lambda)=(\lambda\widetilde{w}_1-\tilde{q}_1)(b_3-b_1)\tilde{r}_0+1$, $\Psi(c+,\lambda)$ 为 $\Psi(c,\lambda)$ 在 c 点处的右极限. 当 $1\leqslant j\leqslant n$ 时有:

$$\Psi(b_{2j+1},\lambda)=\begin{pmatrix}1 & 1 & \tilde{r}_{j-1}\\ 0 & 1 & r_{j-1}\\ \lambda\widetilde{w}_j-\tilde{q}_j & \lambda\widetilde{w}_j-\tilde{q}_j & \tilde{r}_{j-1}(\lambda\widetilde{w}_j-\tilde{q}_j)\end{pmatrix}\Psi(b_{2j-1},\lambda).\tag{16}$$

证明 由于证明方法与引理 2 的证明方法相同,故本文在此省略.

引理 4 令 $\Phi(t,\lambda)=[\phi_{ij}(t,\lambda)]$ 是系统(4)满足初始条件 $\Phi(a,\lambda)=I$ 的基解矩阵,且 $\Psi(t,\lambda)=[\psi_{ij}(t,\lambda)]$ 与引理 3 中给出的意义一致,则有

$$\Phi(b,\lambda)=\Psi(b,\lambda)G\Phi(c,\lambda),\tag{17}$$

其中 $G=[g_{ij}]_{3\times 3}=-D^{-1}C$, $\Psi(c-,\lambda)$ 为 $\Psi(c,\lambda)$ 在 c 点处的左极限.

证明 由转移条件可知 $C\Phi(c-,\lambda)+D\Phi(c+,\lambda)=0$, 从而有 $\Phi(c+,\lambda)=-D^{-1}C\Phi(c-,\lambda)=G\Phi(c-,\lambda)$. 注意到 $\Phi(c,\lambda)=\Phi(c-,\lambda)=\Phi(a_{2m+1},\lambda)$, $\Psi(b,\lambda)=\Phi(b_{2n+1},\lambda)$, $\Phi(c-,\lambda)=I$, 故由引理 2 和引理 3 得 $\Phi(b,\lambda)=\Psi(b,\lambda)G\Phi(c,\lambda)$, 其中 $\Psi(b,\lambda)=\Phi(b_{2n+1},\lambda)$. 证毕.

引理 5　令 $\Phi(t,\lambda)=[\phi_{ij}(t,\lambda)]$ 是系统(4)满足初始条件 $\Phi(a,\lambda)=I$ 的基解矩阵,则对于每一个 $\lambda\in C$,$\Phi(b,\lambda)$ 有:

$$\phi_{11}(b,\lambda)=RR^{*}G^{*}\cdot\prod_{k=0}^{m-1}(\lambda w_k-q_k)\prod_{i=1}^{n-1}(\lambda w_i-q_i)+\phi'_{11}(b,\lambda),$$

$$\phi_{12}(b,\lambda)=RR^{*}G^{*}\cdot\prod_{k=0}^{m-1}(\lambda w_k-q_k)\prod_{i=1}^{n-1}(\lambda w_i-q_i)+\phi'_{12}(b,\lambda),$$

$$\phi_{13}(b,\lambda)=RR^{*}G^{*}\cdot\prod_{k=1}^{m-1}(\lambda w_k-q_k)\prod_{i=1}^{n-1}(\lambda w_i-q_i)+\phi'_{13}(b,\lambda),$$

$$\phi_{21}(b,\lambda)=RR^{*}G^{*}\cdot\prod_{k=0}^{m-1}(\lambda w_k-q_k)\prod_{i=1}^{n-1}(\lambda w_i-q_i)+\phi'_{21}(b,\lambda),$$

$$\phi_{22}(b,\lambda)=RR^{*}G^{*}\cdot\prod_{k=0}^{m-1}(\lambda w_k-q_k)\prod_{i=1}^{n-1}(\lambda w_i-q_i)+\phi'_{22}(b,\lambda),$$

$$\phi_{23}(b,\lambda)=RR^{*}G^{*}\cdot\prod_{k=1}^{m-1}(\lambda w_k-q_k)\prod_{i=1}^{n-1}(\lambda w_i-q_i)+\phi'_{23}(b,\lambda),$$

$$\phi_{31}(b,\lambda)=RR^{*}G^{*}\cdot\prod_{k=0}^{m-1}(\lambda w_k-q_k)\prod_{i=1}^{n-1}(\lambda w_i-q_i)+\phi'_{31}(b,\lambda),$$

$$\phi_{32}(b,\lambda)=RR^{*}G^{*}\cdot\prod_{k=0}^{m-1}(\lambda w_k-q_k)\prod_{i=1}^{n-1}(\lambda w_i-q_i)+\phi'_{32}(b,\lambda),$$

$$\phi_{33}(b,\lambda)=RR^{*}G^{*}\cdot\prod_{k=1}^{m-1}(\lambda w_k-q_k)\prod_{i=1}^{n-1}(\lambda w_i-q_i)+\phi'_{33}(b,\lambda).$$

其中:$G^{*}=g_{11}(\lambda w_0-q_0)+g_{21}(\lambda w_0-q_0)+g_{31}+g_{12}(\lambda w_0-q_0)+g_{22}(\lambda w_0-q_0)+g_{32}+g_{13}(\lambda w_0-q_0)+g_{23}(\lambda w_0-q_0)+g_{33}$, $R=\prod_{k=0}^{m-1}r_k$, $R^{*}=\prod_{i=0}^{n-1}r_i$, $\phi'_{ij}(b,\lambda)=o(RR^{*})$, $i,j=1,2,3$.

证明　由引理 1 可知:

$$\Phi(c,\lambda)=\Phi(a_{2m+1},\lambda)=\begin{pmatrix}1&1&r_{m-1}\\0&1&r_{m-1}\\\lambda w_m-q_m&\lambda w_m-q_m&(\lambda w_m-q_m)r_{m-1}\end{pmatrix}\Phi(a_{2m-1},\lambda)=$$

$$\begin{pmatrix}1&1&r_{m-1}\\0&1&r_{m-1}\\\lambda w_m-q_m&\lambda w_m-q_m&(\lambda w_m-q_m)r_{m-1}\end{pmatrix}\begin{pmatrix}1&1&r_{m-2}\\0&1&r_{m-2}\\\lambda w_{m-1}-q_{m-1}&\lambda w_{m-1}-q_{m-1}&(\lambda w_{m-1}-q_{m-1})r_{m-2}\end{pmatrix}\times$$

$$\Phi(a_{2m-3},\lambda)=\begin{pmatrix}\theta_{11}&\theta_{12}&\theta_{13}\\\theta_{21}&\theta_{22}&\theta_{23}\\\theta_{31}&\theta_{32}&\theta_{33}\end{pmatrix}\Phi(a_{2m-3},\lambda),$$

其中:

$$\theta_{11}=1+r_{m-1}(\lambda w_{m-1}-q_{m-1})=r_{m-1}(\lambda w_{m-1}-q_{m-1})+o(\lambda w_{m-1}-q_{m-1}),$$

$$\theta_{12}=2+r_{m-1}(\lambda w_{m-1}-q_{m-1})=r_{m-1}(\lambda w_{m-1}-q_{m-1})+o(\lambda w_{m-1}-q_{m-1}),$$

$$\theta_{13}=2r_{m-2}+r_{m-1}r_{m-2}(\lambda w_{m-1}-q_{m-1})=r_{m-1}r_{m-2}(\lambda w_{m-1}-q_{m-1})+o(r_{m-1}r_{m-2}(\lambda w_{m-1}-q_{m-1})),$$

$$\theta_{21}=r_{m-1}(\lambda w_{m-1}-q_{m-1})=r_{m-1}(\lambda w_{m-1}-q_{m-1})+o(\lambda w_{m-1}-q_{m-1}),$$

$$\theta_{22}=1+r_{m-1}(\lambda w_{m-1}-q_{m-1})=r_{m-1}(\lambda w_{m-1}-q_{m-1})+o(r_{m-1}(\lambda w_{m-1}-q_{m-1})),$$

$$\theta_{23}=r_{m-2}+r_{m-1}r_{m-2}(\lambda w_{m-1}-q_{m-1})=r_{m-1}r_{m-2}(\lambda w_{m-1}-q_{m-1})+o(r_{m-1}r_{m-2}(\lambda w_{m-1}-q_{m-1})),$$

$$\theta_{31}=\lambda w_m-q_m+r_{m-1}(\lambda w_m-q_m)(\lambda w_{m-1}-q_{m-1})=$$
$$r_{m-1}(\lambda w_m-q_m)(\lambda w_{m-1}-q_{m-1})+o(r_{m-1}(\lambda w_m-q_m)(\lambda w_{m-1}-q_{m-1})),$$

$$\theta_{32}=2(\lambda w_m-q_m)+r_{m-1}(\lambda w_m-q_m)(\lambda w_{m-1}-q_{m-1})=$$

$$r_{m-1}(\lambda w_m - q_m)(\lambda w_{m-1} - q_{m-1}) + o(r_{m-1}(\lambda w_m - q_m)(\lambda w_{m-1} - q_{m-1})),$$

$$\theta_{33} = 2r_{m-2}(\lambda w_m - q_m) + r_{m-1}r_{m-2}(\lambda w_m - q_m)(\lambda w_{m-1} - q_{m-1}) =$$
$$r_{m-1}r_{m-2}(\lambda w_m - q_m)(\lambda w_{m-1} - q_{m-1}) + o(r_{m-1}r_{m-2}(\lambda w_m - q_m)(\lambda w_{m-1} - q_{m-1})).$$

由于 $\Phi(a_{2m-3},\lambda) = \begin{pmatrix} 1 & 1 & r_{m-3} \\ 0 & 1 & r_{m-3} \\ \lambda w_{m-2} - q_{m-2} & \lambda w_{m-2} - q_{m-2} & (\lambda w_{m-2} - q_{m-2})r_{m-3} \end{pmatrix} \Phi(a_{2m-5},\lambda)$，故 $\Phi(c,\lambda) =$

$$\begin{pmatrix} \theta_{11} & \theta_{12} & \theta_{13} \\ \theta_{21} & \theta_{22} & \theta_{23} \\ \theta_{31} & \theta_{32} & \theta_{33} \end{pmatrix} \Phi(a_{2m-3},\lambda) = \begin{pmatrix} \theta_{11} & \theta_{12} & \theta_{13} \\ \theta_{21} & \theta_{22} & \theta_{23} \\ \theta_{31} & \theta_{32} & \theta_{33} \end{pmatrix} \begin{pmatrix} 1 & 1 & r_{m-3} \\ 0 & 1 & r_{m-3} \\ \lambda w_{m-2} - q_{m-2} & \lambda w_{m-2} - q_{m-2} & (\lambda w_{m-2} - q_{m-2})r_{m-3} \end{pmatrix} \times$$

$\Phi(a_{2m-5},\lambda) = \begin{pmatrix} \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{31} & \eta_{32} & \eta_{33} \end{pmatrix} \Phi(a_{2m-5},\lambda)$，其中：

$$\eta_{11} = \theta_{11} + (\lambda w_{m-2} - q_{m-2})\theta_{13} = r_{m-1}r_{m-2}(\lambda w_{m-1} - q_{m-1})(\lambda w_{m-2} - q_{m-2}) +$$
$$o(r_{m-1}r_{m-2}(\lambda w_{m-1} - q_{m-1})(\lambda w_{m-2} - q_{m-2})),$$

$$\eta_{12} = \theta_{11} + \theta_{12} + (\lambda w_{m-2} - q_{m-2})\theta_{13} = r_{m-1}r_{m-2}(\lambda w_{m-1} - q_{m-1})(\lambda w_{m-2} - q_{m-2}) +$$
$$o(r_{m-1}r_{m-2}(\lambda w_{m-1} - q_{m-1})(\lambda w_{m-2} - q_{m-2})),$$

$$\eta_{13} = r_{m-3}\theta_{11} + r_{m-3}\theta_{12} + r_{m-3}(\lambda w_{m-2} - q_{m-2})\theta_{13} = r_{m-1}r_{m-2}r_{m-3}(\lambda w_{m-1} - q_{m-1})(\lambda w_{m-2} - q_{m-2}) +$$
$$o(r_{m-1}r_{m-2}r_{m-3}(\lambda w_{m-1} - q_{m-1})(\lambda w_{m-2} - q_{m-2})),$$

$$\eta_{21} = \theta_{21} + (\lambda w_{m-2} - q_{m-2})\theta_{23} = r_{m-1}r_{m-2}(\lambda w_{m-1} - q_{m-1})(\lambda w_{m-2} - q_{m-2}) +$$
$$o(r_{m-1}r_{m-2}(\lambda w_{m-1} - q_{m-1})(\lambda w_{m-2} - q_{m-2})),$$

$$\eta_{22} = \theta_{21} + \theta_{22} + (\lambda w_{m-2} - q_{m-2})\theta_{23} = r_{m-1}r_{m-2}(\lambda w_{m-1} - q_{m-1})(\lambda w_{m-2} - q_{m-2}) +$$
$$o(r_{m-1}r_{m-2}(\lambda w_{m-1} - q_{m-1}))((\lambda w_{m-2} - q_{m-2})),$$

$$\eta_{23} = r_{m-3}\theta_{21} + r_{m-3}\theta_{22} + r_{m-3}(\lambda w_{m-2} - q_{m-2})\theta_{23} =$$
$$r_{m-1}r_{m-2}r_{m-3}(\lambda w_{m-1} - q_{m-1})(\lambda w_{m-2} - q_{m-2}) + o(r_{m-1}r_{m-2}r_{m-3}(\lambda w_{m-1} - q_{m-1})(\lambda w_{m-2} - q_{m-2})),$$

$$\eta_{31} = \theta_{31} + (\lambda w_{m-2} - q_{m-2})\theta_{33} = r_{m-1}r_{m-2}(\lambda w_m - q_m)(\lambda w_{m-1} - q_{m-1})(\lambda w_{m-2} - q_{m-2}) +$$
$$o(r_{m-1}r_{m-2}(\lambda w_m - q_m)(\lambda w_{m-1} - q_{m-1})(\lambda w_{m-2} - q_{m-2})),$$

$$\eta_{32} = \theta_{31} + \theta_{32} + (\lambda w_{m-2} - q_{m-2})\theta_{33} = r_{m-1}r_{m-2}(\lambda w_m - q_m)(\lambda w_{m-1} - q_{m-1})(\lambda w_{m-2} - q_{m-2}) +$$
$$o(r_{m-1}r_{m-2}(\lambda w_m - q_m)(\lambda w_{m-1} - q_{m-1})(\lambda w_{m-2} - q_{m-2})),$$

$$\eta_{33} = r_{m-3}\theta_{31} + r_{m-3}\theta_{32} + r_{m-3}(\lambda w_{m-2} - q_{m-2})\theta_{33} = r_{m-1}r_{m-2}r_{m-3}(\lambda w_m - q_m)(\lambda w_{m-1} - q_{m-1}) \cdot$$
$$(\lambda w_{m-2} - q_{m-2}) + o(r_{m-1}r_{m-2}r_{m-3}(\lambda w_m - q_m)(\lambda w_{m-1} - q_{m-1})(\lambda w_{m-2} - q_{m-2})).$$

对上述计算结果进行递推可得 $\Phi(c,\lambda) = \begin{pmatrix} \xi_{11} & \xi_{12} & \xi_{13} \\ \xi_{21} & \xi_{22} & \xi_{23} \\ \xi_{31} & \xi_{32} & \xi_{33} \end{pmatrix} \Phi(a_1,\lambda)$，其中：

$$\xi_{11} = \prod_{k=1}^{m-1} r_k \prod_{k=1}^{m-1} (\lambda w_k - q_k) + o\left(\prod_{k=1}^{m-1} r_k \prod_{k=1}^{m-1} (\lambda w_k - q_k)\right),$$

$$\xi_{12} = \prod_{k=1}^{m-1} r_k \prod_{k=1}^{m-1} (\lambda w_k - q_k) + o\left(\prod_{k=1}^{m-1} r_k \prod_{k=1}^{m-1} (\lambda w_k - q_k)\right),$$

$$\xi_{13} = \prod_{k=0}^{m-1} r_k \prod_{k=1}^{m-1} (\lambda w_k - q_k) + o\left(\prod_{k=1}^{m-1} r_k \prod_{k=1}^{m-1} (\lambda w_k - q_k)\right),$$

$$\xi_{21} = \prod_{k=1}^{m-1} r_k \prod_{k=1}^{m-1} (\lambda w_k - q_k) + o\left(\prod_{k=1}^{m-1} r_k \prod_{k=1}^{m-1} (\lambda w_k - q_k)\right),$$

$$\xi_{22}=\prod_{k=1}^{m-1}r_k\prod_{k=1}^{m-1}(\lambda w_k-q_k)+o\Big(\prod_{k=1}^{m-1}r_k\prod_{k=1}^{m-1}(\lambda w_k-q_k)\Big),$$
$$\xi_{23}=\prod_{k=0}^{m-1}r_k\prod_{k=1}^{m-1}(\lambda w_k-q_k)+o\Big(\prod_{k=1}^{m-1}r_k\prod_{k=1}^{m-1}(\lambda w_k-q_k)\Big),$$
$$\xi_{31}=\prod_{k=1}^{m-1}r_k\prod_{k=1}^{m-1}(\lambda w_k-q_k)+o\Big(\prod_{k=1}^{m-1}r_k\prod_{k=1}^{m-1}(\lambda w_k-q_k)\Big),$$
$$\xi_{32}=\prod_{k=1}^{m-1}r_k\prod_{k=1}^{m-1}(\lambda w_k-q_k)+o\Big(\prod_{k=1}^{m-1}r_k\prod_{k=1}^{m-1}(\lambda w_k-q_k)\Big),$$
$$\xi_{33}=\prod_{k=0}^{m-1}r_k\prod_{k=1}^{m-1}(\lambda w_k-q_k)+o\Big(\prod_{k=1}^{m-1}r_k\prod_{k=1}^{m-1}(\lambda w_k-q_k)\Big).$$

又因 $\Phi(a_1,\lambda)=\begin{pmatrix}1 & a_1-a_0 & 0\\ 0 & 1 & 0\\ \lambda w_0-q_0 & (\lambda w_0-q_0)(a_1-a_0) & 1\end{pmatrix}$,故有 $\Phi(c,\lambda)=\begin{pmatrix}\xi_{11} & \xi_{12} & \xi_{13}\\ \xi_{21} & \xi_{22} & \xi_{23}\\ \xi_{31} & \xi_{32} & \xi_{33}\end{pmatrix}\Phi(a_1,\lambda)=$

$\begin{pmatrix}\xi_{11} & \xi_{12} & \xi_{13}\\ \xi_{21} & \xi_{22} & \xi_{23}\\ \xi_{31} & \xi_{32} & \xi_{33}\end{pmatrix}\begin{pmatrix}1 & a_1-a_0 & 0\\ 0 & 1 & 0\\ \lambda w_0-q_0 & (\lambda w_0-q_0)(a_1-a_0) & 1\end{pmatrix}=\begin{pmatrix}\phi_{11}(c,\lambda) & \phi_{12}(c,\lambda) & \phi_{13}(c,\lambda)\\ \phi_{21}(c,\lambda) & \phi_{22}(c,\lambda) & \phi_{23}(c,\lambda)\\ \phi_{31}(c,\lambda) & \phi_{32}(c,\lambda) & \phi_{33}(c,\lambda)\end{pmatrix}$,即:

$$\begin{aligned}
\phi_{11}(c,\lambda)&=\prod_{k=0}^{m-1}r_k\prod_{k=0}^{m-1}(\lambda w_k-q_k)+o\Big(\prod_{k=0}^{m_0-1}r_k\prod_{k=0}^{m_0-1}(q_k-\lambda w_k)\Big),\\
\phi_{12}(c,\lambda)&=\prod_{k=0}^{m-1}r_k\prod_{k=0}^{m-1}(\lambda w_k-q_k)+o\Big(\prod_{k=1}^{m_0-1}r_k\prod_{k=1}^{m_0-1}(q_k-\lambda w_k)\Big),\\
\phi_{13}(c,\lambda)&=\prod_{k=0}^{m-1}r_k\prod_{k=1}^{m-1}(\lambda w_k-q_k)+o\Big(\prod_{k=0}^{m_0-1}r_k\prod_{k=0}^{m_0-1}(q_k-\lambda w_k)\Big),\\
\phi_{21}(c,\lambda)&=\prod_{k=0}^{m-1}r_k\prod_{k=0}^{m-1}(\lambda w_k-q_k)+o\Big(\prod_{k=1}^{m_0-1}r_k\prod_{k=1}^{m_0-1}(q_k-\lambda w_k)\Big),\\
\phi_{22}(c,\lambda)&=\prod_{k=0}^{m-1}r_k\prod_{k=0}^{m-1}(\lambda w_k-q_k)+o\Big(\prod_{k=0}^{m_0-1}r_k\prod_{k=0}^{m_0-1}(q_k-\lambda w_k)\Big),\\
\phi_{23}(c,\lambda)&=\prod_{k=0}^{m-1}r_k\prod_{k=1}^{m-1}(\lambda w_k-q_k)+o\Big(\prod_{k=1}^{m_0-1}r_k\prod_{k=0}^{m_0-1}(q_k-\lambda w_k)\Big),\\
\phi_{31}(c,\lambda)&=\prod_{k=0}^{m-1}r_k\prod_{k=0}^{m-1}(\lambda w_k-q_k)+o\Big(\prod_{k=0}^{m_0-1}r_k\prod_{k=0}^{m_0-1}(q_k-\lambda w_k)\Big),\\
\phi_{32}(c,\lambda)&=\prod_{k=0}^{m-1}r_k\prod_{k=0}^{m-1}(\lambda w_k-q_k)+o\Big(\prod_{k=1}^{m_0-1}r_k\prod_{k=0}^{m_0-1}(q_k-\lambda w_k)\Big),\\
\phi_{33}(c,\lambda)&=\prod_{k=0}^{m-1}r_k\prod_{k=1}^{m-1}(\lambda w_k-q_k)+o\Big(\prod_{k=0}^{m_0-1}r_k\prod_{k=0}^{m_0-1}(q_k-\lambda w_k)\Big).
\end{aligned}\tag{18}$$

再结合引理 4 中的结果($\Phi(b,\lambda)=\Psi(b,\lambda)G\Phi(c,\lambda)$)可得:

$$\psi_{11}(b,\lambda)=\prod_{i=0}^{n-1}r_i\prod_{i=0}^{n-1}(\lambda w_i-q_i)+o\Big(\prod_{i=0}^{n-1}r_i\prod_{i=0}^{n-1}(\lambda w_i-q_i)\Big),$$
$$\psi_{12}(b,\lambda)=\prod_{i=0}^{n-1}r_i\prod_{i=0}^{n-1}(\lambda w_i-q_i)+o\Big(\prod_{i=0}^{n-1}r_i\prod_{i=0}^{n-1}(\lambda w_i-q_i)\Big),$$
$$\psi_{13}(b,\lambda)=\prod_{i=0}^{n-1}r_i\prod_{i=1}^{n-1}(\lambda w_i-q_i)+o\Big(\prod_{i=0}^{n-1}r_i\prod_{i=1}^{n-1}(\lambda w_i-q_i)\Big),$$
$$\psi_{21}(b,\lambda)=\prod_{i=0}^{n-1}r_i\prod_{i=0}^{n-1}(\lambda w_i-q_i)+o\Big(\prod_{i=0}^{n-1}r_i\prod_{i=0}^{n-1}(\lambda w_i-q_i)\Big),$$

$$\psi_{22}(b,\lambda)=\prod_{i=0}^{n-1}r_i\prod_{i=0}^{n-1}(\lambda w_i-q_i)+o\Big(\prod_{i=0}^{n-1}r_i\prod_{i=0}^{n-1}(\lambda w_i-q_i)\Big),\tag{19}$$
$$\psi_{23}(b,\lambda)=\prod_{i=0}^{n-1}r_i\prod_{i=1}^{n-1}(\lambda w_i-q_i)+o\Big(\prod_{i=0}^{n-1}r_i\prod_{i=1}^{n-1}(\lambda w_i-q_i)\Big),$$
$$\psi_{31}(b,\lambda)=\prod_{i=0}^{n-1}r_i\prod_{i=0}^{n-1}(\lambda w_i-q_i)+o\Big(\prod_{i=0}^{n-1}r_i\prod_{i=0}^{n-1}(\lambda w_i-q_i)\Big),$$
$$\psi_{32}(b,\lambda)=\prod_{i=0}^{n-1}r_i\prod_{i=0}^{n-1}(\lambda w_i-q_i)+o\Big(\prod_{i=0}^{n-1}r_i\prod_{i=0}^{n-1}(\lambda w_i-q_i)\Big),$$
$$\psi_{33}(b,\lambda)=\prod_{i=0}^{n-1}r_i\prod_{i=1}^{n-1}(\lambda w_i-q_i)+o\Big(\prod_{i=0}^{n-1}r_i\prod_{i=1}^{n-1}(\lambda w_i-q_i)\Big).$$

又因为$G=\begin{pmatrix}g_{11}&g_{12}&g_{13}\\g_{21}&g_{22}&g_{23}\\g_{31}&g_{32}&g_{33}\end{pmatrix}$,于是由引理4可知$\Phi(b,\lambda)=\Psi(b,\lambda)G\Phi(c,\lambda)$. 再结合式(18)和(19)可得:

$$\phi_{11}(b,\lambda)=RR^*G^*\cdot\prod_{k=0}^{m-1}(\lambda w_k-q_k)\prod_{i=1}^{n-1}(\lambda w_i-q_i)+\phi'_{11}(b,\lambda),$$
$$\phi_{12}(b,\lambda)=RR^*G^*\cdot\prod_{k=0}^{m-1}(\lambda w_k-q_k)\prod_{i=1}^{n-1}(\lambda w_i-q_i)+\phi'_{12}(b,\lambda),$$
$$\phi_{13}(b,\lambda)=RR^*G^*\cdot\prod_{k=1}^{m-1}(\lambda w_k-q_k)\prod_{i=1}^{n-1}(\lambda w_i-q_i)+\phi'_{13}(b,\lambda),$$
$$\phi_{21}(b,\lambda)=RR^*G^*\cdot\prod_{k=0}^{m-1}(\lambda w_k-q_k)\prod_{i=1}^{n-1}(\lambda w_i-q_i)+\phi'_{21}(b,\lambda),$$
$$\phi_{22}(b,\lambda)=RR^*G^*\cdot\prod_{k=0}^{m-1}(\lambda w_k-q_k)\prod_{i=1}^{n-1}(\lambda w_i-q_i)+\phi'_{22}(b,\lambda),$$
$$\phi_{23}(b,\lambda)=RR^*G^*\cdot\prod_{k=1}^{m-1}(\lambda w_k-q_k)\prod_{i=1}^{n-1}(\lambda w_i-q_i)+\phi'_{23}(b,\lambda),$$
$$\phi_{31}(b,\lambda)=RR^*G^*\cdot\prod_{k=0}^{m-1}(\lambda w_k-q_k)\prod_{i=1}^{n-1}(\lambda w_i-q_i)+\phi'_{31}(b,\lambda),$$
$$\phi_{32}(b,\lambda)=RR^*G^*\cdot\prod_{k=0}^{m-1}(\lambda w_k-q_k)\prod_{i=1}^{n-1}(\lambda w_i-q_i)+\phi'_{32}(b,\lambda),$$
$$\phi_{33}(b,\lambda)=RR^*G^*\cdot\prod_{k=1}^{m-1}(\lambda w_k-q_k)\prod_{i=1}^{n-1}(\lambda w_i-q_i)+\phi'_{33}(b,\lambda),$$

其中$G^*=g_{11}(\lambda w_0-q_0)+g_{21}(\lambda w_0-q_0)+g_{31}+g_{12}(\lambda w_0-q_0)+g_{22}(\lambda w_0-q_0)+g_{32}+g_{13}(\lambda w_0-q_0)+g_{23}(\lambda w_0-q_0)+g_{33}$, $R=\prod_{k=0}^{m-1}r_k$, $R^*=\prod_{i=0}^{n-1}r_i$, $\phi'_{ij}(b,\lambda)=o(RR^*)$, $i,j=1,2,3$. 引理5得证.

定理1 设$m,n\in N$, $g_{12}\neq 0$,且式(8)—(10)成立,$H(\lambda)=(h_{ij}(\lambda))_{3\times 3}$与引理1中的定义一致,则问题(1)至多有$m+n+2$个特征值.

证明 由引理1可知,$\Delta(\lambda)=\sum_{i=1,j=1}^{3}h_{ij}(\lambda)\phi_{ij}(b,\lambda)$. 再由引理5可知,$\phi_{11}(b,\lambda)$、$\phi_{12}(b,\lambda)$、$\phi_{13}(b,\lambda)$、$\phi_{21}(b,\lambda)$、$\phi_{22}(b,\lambda)$、$\phi_{23}(b,\lambda)$、$\phi_{31}(b,\lambda)$、$\phi_{32}(b,\lambda)$、$\phi_{33}(b,\lambda)$关于$\lambda$的次数分别为$m+n$、$m+n$、$m+n-1$、$m+n$、$m+n$、$m+n-1$、$m+n$、$m+n$、$m+n-1$. 当$h_{11}\neq 0$时,$\Delta(\lambda)$的次数为$m+n+2$. 再由代数学的基本定理可知,$\Delta(\lambda)=0$时,至多有$m+n+2$个特征值. 定理1得证.

下面考虑区别于式(1)的另一类情形,即:

$$\begin{cases}(py')''+qy=\lambda wy,\\ A_\lambda Y(a)+B_\lambda Y(b)=0,\\ CY(c-)+DY(c+)=0.\end{cases}$$

由式(1) 和式(2) 可知，两类含谱参数问题的区别仅是对方程中 py'' 和 py' 的求导不同，即式(1) 中是对 py'' 求一阶导数，而式(2) 中是对 py' 求二阶导数，因此在证明过程中仅给出各自方程等价系统的矩阵形式即可. 故令 $u_1=y$，$u_2=y'$，$u_3=py''$，由此可得与方程$(py'')'+qy=\lambda wy$ 和$(py')''+qy=\lambda wy$ 等价的系统为：

$$u_1'=ru_2,\ u_2'=u_3,\ u_3'=(\lambda w-q)u_1. \tag{20}$$

由于证明式(2) 的方法与证明式(1) 的方法相似，故本文在此省略. 其中的区别是：在系统(20) 中，u_1 在 r 恒等于零的子区间上是常数，u_3 在 q 和 w 恒等于零的子区间上是常数.

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