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# 分数阶中立型随机时滞微分方程的 波形松弛方法

李佳敏, 丁小丽, 王苗苗  
(西安工程大学 理学院, 西安 710600)

**摘要:** 针对大多数分数阶中立型随机时滞微分方程无法给出精确解的问题, 给出了方程的一种数值解法. 该方法首先将波形松弛方法推广到具有常延迟项的分数阶中立型随机微分方程, 然后在分裂函数满足 Lipschitz 条件下证明了波形松弛方法在均方意义下收敛. 数值模拟表明, 波形松弛方法可用于求解分数阶中立型随机时滞微分方程.

**关键词:** 分数阶中立型随机时滞微分方程; 波形松弛法; 收敛性分析; 数值模拟

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## Waveform relaxation method for fractional neutral stochastic delay differential equations

LI Jiamin, DING Xiaoli, WANG Miaomiao  
(School of Science, Xi'an Polytechnic University, Xi'an 710600, China)

**Abstract:** In order to solve the problem that most fractional neutral stochastic delay differential equations cannot give exact solutions, a numerical method is presented. The waveform relaxation method is extended to fractional neutral stochastic differential equations with constant delay terms, and then it is proved that the waveform relaxation method converges in the mean square sense under the Lipschitz condition of splitting function. Numerical simulation shows that the waveform relaxation method can be used to solve fractional neutral stochastic delay differential equations.

**Keywords:** fractional neutral stochastic delay differential equations; waveform relaxation method; convergence analysis; numerical simulation

## 0 引言

分数阶中立型泛函微分方程由于可以模拟较为复杂的动力系统, 因此受到学者们的重视. 2014 年, 张艳敏<sup>[1]</sup>讨论了时间分数阶中立型时滞微分方程, 并给出了该方程的数值解. 2016 年, 杨水平<sup>[2]</sup>讨论了一类分数阶线性中立型延迟微分方程的初值问题, 并给出了其解渐进稳定的充要条件. 2017 年, 张玉峰等<sup>[3]</sup>研究了分数阶中立型时滞微分方程在 Caputo 导数意义下其解的存在唯一问题以及其通解表达式. 2019 年, Ding 等<sup>[4]</sup>利用波形松弛方法给出了一类具有几乎扇形算子的分数阶随机演化方程的数值解.

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第一作者: 李佳敏(1983—), 女, 硕士研究生, 研究方向为微分方程的基本理论、数值计算方法及其应用.

通信作者: 丁小丽(1983—), 女, 博士, 教授, 研究方向为微分方程的基本理论、数值计算方法及其应用.

2021年,王子丰等<sup>[5]</sup>利用截断 Euler-Maruyama 法研究了中立型随机泛函微分方程的数值解.基于上述研究,本文研究一类分数阶中立型随机时滞微分方程的波形松弛方法,并给出了分数阶中立型随机时滞微分方程的波形松弛格式,以及该方法的收敛结果.

## 1 预备知识

给定一个完备的概率空间  $\{\Omega, \mathcal{F}, P\}$ , 其中  $\Omega = [0, T]$  为实轴上的有限区间,  $T$  是正常数,  $\mathcal{F}$  是  $\Omega$  的滤子,  $P$  是概率测度;  $C([0, T]; \mathbf{R}^n)$  是由所有的连续函数  $f$  构成的 Banach 空间, 其中  $\|f\| = \max_{t \in [0, T]} \|f(t)\|$ ;  $L^q(\Omega; \mathbf{R}^n)$ ,  $q \geq 1$  是由所有的可积函数  $f$  构成的 Banach 空间, 其中  $\|f\|_2 = \left( \int_0^T |f(t)|^2 dt \right)^{\frac{1}{2}}$ .

**定义 1<sup>[4]</sup>** 设  $0 < \alpha \leq 1$ ,  $t \in \Omega$ , 定义函数  $f(t) \in L^1(\Omega; \mathbf{R}^n)$  的  $\alpha$  阶 Riemann-Liouville 型分数阶积分为  $(I_{0+}^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds$ ,  $t > 0$ , 其中  $\Gamma(\cdot)$  是 Gamma 函数,  $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ .

**定义 2<sup>[6]</sup>** 两个算子  $\mathcal{T}_1, \mathcal{T}_2$  分别为  $(\mathcal{T}_1 \phi)(t) = \int_0^t \phi(s) ds$ ,  $(\mathcal{T}_2 \phi)(t) = \int_0^t (t-s)^{\alpha-1} \phi(s) ds$ , 其中  $0 \leq \alpha \leq 1$ ,  $\phi(t) \in C(\Omega; \mathbf{R}^n)$ .

**引理 1<sup>[6]</sup>** 设  $0 \leq \alpha \leq 1$ ,  $\phi(t) \in C(\Omega; \mathbf{R}^n)$ , 则  $(\mathcal{T}_1, \mathcal{T}_2 \phi)(t) = (\mathcal{T}_2, \mathcal{T}_1 \phi)(t)$ .

**引理 2<sup>[6]</sup>** 设  $0 \leq \alpha \leq 1$ ,  $\phi(t) \in C(\Omega; \mathbf{R}^n)$ , 且当  $i = 1, 2, 3, \dots$  时以下关系成立:

$$(\mathcal{T}_1^i \phi)(t) = \frac{1}{\Gamma(i)} \int_0^t (t-\delta)^{i-1} \phi(\delta) d\delta, \quad (\mathcal{T}_2^i \phi)(t) = \frac{\Gamma(\alpha)}{\Gamma(i\alpha)} \int_0^t (t-\delta)^{i\alpha-1} \phi(\delta) d\delta.$$

**引理 3<sup>[4]</sup>** 设  $0 < \alpha < 1$ ,  $a(t)$  是在时间区间  $\Omega$  上的一个局部可积的非负函数,  $b(t)$  和  $g(t)$  是在  $\Omega$  上的非负非减且有界的连续函数. 如果  $v(t)$  为非负, 在  $\Omega$  上局部可积, 且满足  $v(t) \leq a(t) + b(t) \times \int_0^t v(s) ds + g(t) \int_0^t (t-s)^{\alpha-1} v(s) ds$ , 则:

$$v(t) \leq a(t) + \sum_{n=1}^{\infty} \sum_{i=0}^n \binom{n}{i} b^{n-i}(t) g^i(t) \frac{(\Gamma(\alpha))^i}{\Gamma(i\alpha + n - i)} \int_0^t (t-s)^{i\alpha - (i+1-n)} a(s) ds.$$

**推论 1<sup>[4]</sup>** 假设  $v(t)$  满足引理 2 中的条件, 且  $a(t)$  在  $\Omega$  上非减, 则  $v(t) \leq a(t) E_\alpha(g(t) \Gamma(\alpha) t^\alpha) \cdot \exp\left\{\frac{1}{\alpha} b(t) t\right\}$ , 其中  $E_\alpha$  为 Mittag-Leffler 函数,  $E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + 1)}$ .

**引理 4<sup>[7]</sup>** 算子  $\mathcal{T}_1, \mathcal{T}_2$  在空间  $C(\Omega; \mathbf{R}^n)$  上是紧算子, 且  $\sigma(\mathcal{T}_1) = \sigma(\mathcal{T}_2) = \{0\}$ , 其中  $\sigma(\cdot)$  为算子的谱. 考虑如下分数阶中立型随机时滞微分方程的初值问题:

$$\begin{cases} d[x(t) - D(x(t-\tau))] = f(x(t), x(t-\tau), t) dt + g(x(t), x(t-\tau), t) d\mathbf{B}(t) + \\ \quad \Gamma(\alpha + 1) dI_{0+}^\alpha (\sigma(x(t), x(t-\tau), t)), t \in [0, T]; \\ x(t) = \zeta(t), t \in [-\tau, 0]. \end{cases} \quad (1)$$

其中:  $\tau$  为给定的时滞量,  $\tau > 0$ ;  $f, \sigma \in C(\mathbf{R}^n \times \mathbf{R}^n \times [0, T]; \mathbf{R}^n)$ ;  $g \in C(\mathbf{R}^n \times \mathbf{R}^n \times [0, T]; \mathbf{R}^{n \times m})$ ;  $D \in C(\mathbf{R}^n; \mathbf{R}^n)$  为中立项;  $\mathbf{B}(t) = (B_1(t), \dots, B_m(t))^T$  是在完备的概率空间  $\{\Omega, \mathcal{F}, P\}$  上的  $m$  维布朗运动;  $\zeta \in C([-\tau, 0]; \mathbf{R}^n)$ , 且满足  $E|\zeta(t)|^2 < \infty$ .

## 2 波形松弛方法的迭代格式及其收敛性分析

分数阶中立型随机时滞微分方程(1)的波形松弛方法为:

$$\begin{cases} d[x^{k+1}(t) - D(x^{k+1}(t-\tau))] = F(x^{k+1}(t), x^k(t), x^{k+1}(t-\tau), x^k(t-\tau), t) dt + \\ \quad G(x^{k+1}(t), x^k(t), x^{k+1}(t-\tau), x^k(t-\tau), t) d\mathbf{B}(t) + \\ \quad \Gamma(\alpha+1) dI_{0+}^{\alpha}(H(x^{k+1}(t), x^k(t), x^{k+1}(t-\tau), x^k(t-\tau), t)), t \in [0, T]; \\ x^{k+1}(t) = \zeta(t), t \in [-\tau, 0]. \end{cases} \quad (2)$$

其中分裂函数  $F: \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \times [0, T] \rightarrow \mathbf{R}^n$ ,  $G: \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \times [0, T] \rightarrow \mathbf{R}^{n \times m}$ ,  $H: \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \times [0, T] \rightarrow \mathbf{R}^n$  满足:

$$F(x(t), x(t), x(t-\tau), x(t-\tau), t) = f(x(t), x(t-\tau), t),$$

$$G(x(t), x(t), x(t-\tau), x(t-\tau), t) = g(x(t), x(t-\tau), t),$$

$$H(x(t), x(t), x(t-\tau), x(t-\tau), t) = \sigma(x(t), x(t-\tau), t).$$

波形松弛方法的初始迭代函数满足  $x^{(0)}(t) = \zeta(t)$ ,  $t \in [-\tau, 0]$ ;  $x^{(0)}(t) = \zeta(0)$ ,  $t \in [0, T]$ .

**假设 1** 假设  $M \in (0, 1)$ , 则对所有的  $\phi, \varphi \in C([-\tau, 0]; \mathbf{R}^n)$  有  $|D(\phi) - D(\varphi)| \leq M|\phi - \varphi|$ .

**假设 2** 对任意的  $t \in [0, T]$ ,  $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4 \in \mathbf{R}^n$ , 存在非负常数  $L_1, L_2$  和  $L_3$ , 并使得:

$$\begin{aligned} & |F(x_1, x_2, x_3, x_4, t) - F(y_1, y_2, y_3, y_4, t)| \leq \\ & L_1(|x_1 - y_1|^2 + |x_2 - y_2|^2 + |x_3 - y_3|^2 + |x_4 - y_4|^2), \\ & |G(x_1, x_2, x_3, x_4, t) - G(y_1, y_2, y_3, y_4, t)| \leq \\ & L_2(|x_1 - y_1|^2 + |x_2 - y_2|^2 + |x_3 - y_3|^2 + |x_4 - y_4|^2), \\ & |H(x_1, x_2, x_3, x_4, t) - H(y_1, y_2, y_3, y_4, t)| \leq \\ & L_3(|x_1 - y_1|^2 + |x_2 - y_2|^2 + |x_3 - y_3|^2 + |x_4 - y_4|^2). \end{aligned}$$

**定理 1** 设  $0 \leq \alpha \leq 1$ , 如果分裂函数  $F, G$  和  $H$  满足假设 2, 则在均方意义下由波形松弛方法(2)产生的序列  $(\{x^k(t)\}_{-\tau \leq t \leq T}, k=0, 1, \dots)$  一致收敛于方程(1)的解  $\{x(t)\}_{-\tau \leq t \leq T}$ .

**证明** 为了证明方便, 定义:

$$F(t) = F(x(t), x(t), x(t-\tau), x(t-\tau), t),$$

$$G(t) = G(x(t), x(t), x(t-\tau), x(t-\tau), t),$$

$$H(t) = H(x(t), x(t), x(t-\tau), x(t-\tau), t),$$

$$F^k(t) = F(x^{k+1}(t), x^k(t), x^{k+1}(t-\tau), x^k(t-\tau), t),$$

$$G^k(t) = G(x^{k+1}(t), x^k(t), x^{k+1}(t-\tau), x^k(t-\tau), t),$$

$$H^k(t) = H(x^{k+1}(t), x^k(t), x^{k+1}(t-\tau), x^k(t-\tau), t).$$

根据以上定义, 方程(1)和(2)可被写为:

$$\begin{cases} x(t) = \zeta(0) + D(x(t-\tau)) - D(x(-\tau)) + \int_0^t F(s) ds + \int_0^t G(s) d\mathbf{B}(s) + \\ \quad \alpha \int_0^t (t-s)^{\alpha-1} H(s) ds, t \in [0, T]; \\ x(t) = \zeta(t), t \in [-\tau, 0]. \end{cases} \quad (3)$$

$$\begin{cases} x^{k+1}(t) = \zeta(0) + D(x^{k+1}(t-\tau)) - D(x(-\tau)) + \int_0^t F^k(s) ds + \int_0^t G^k(s) d\mathbf{B}(s) + \\ \quad \alpha \int_0^t (t-s)^{\alpha-1} H^k(s) ds, t \in [0, T]; \\ x^{k+1}(t) = \zeta(t), t \in [-\tau, 0]. \end{cases} \quad (4)$$

当  $t \in [0, \tau]$  时, 将式(4)减去式(3), 且令  $e^k(t) = x^k(t) - x(t)$ , 则可得:

$$\begin{cases} e^{k+1}(t) = D(x^{k+1}(t-\tau)) - D(x(t-\tau)) + \int_0^t (F^k(s) - F(s)) ds + \\ \quad \int_0^t (G^k(s) - G(s)) d\mathbf{B}(s) + \alpha \int_0^t (t-s)^{\alpha-1} (H^k(s) - H(s)) ds, t \in [0, T]; \\ e^{k+1}(t) = 0, t \in [-\tau, 0]. \end{cases} \quad (5)$$

于是利用假设 1、不等式  $|a+b+c+e| \leq 4|a|^2 + 4|b|^2 + 4|c|^2 + 4|e|^2$  以及 Hölder 不等式可得:

$$\begin{aligned} |e^{k+1}(t)|^2 &\leq 4|D(x^{k+1}(t-\tau)) - D(x(t-\tau))|^2 + 4\left|\int_0^t (F^k(s) - F(s))ds\right|^2 + \\ &4\left|\int_0^t (G^k(s) - G(s))d\mathbf{B}(s)\right|^2 + 4\left|\int_0^t (t-s)^{\alpha-1}(H^k(s) - H(s))ds\right|^2 \leq \\ &4M^2|x^{k+1}(t-\tau) - x(t-\tau)|^2 + 4t\int_0^t |F^k(s) - F(s)|^2 ds + \\ &4\left|\int_0^t (G^k(s) - G(s))d\mathbf{B}(s)\right|^2 + 4\alpha t^\alpha \int_0^t (t-s)^{\alpha-1} |H^k(s) - H(s)|^2 ds. \end{aligned}$$

在上式两边同时取期望并利用假设 2 以及 Itô 积分的等距性可得:

$$\begin{aligned} E|e^{k+1}(t)|^2 &\leq 4M^2 E|x^{k+1}(t-\tau) - x(t-\tau)|^2 + 4t\int_0^t E|F^k(s) - F(s)|^2 ds + \\ &4\int_0^t E|G^k(s) - G(s)|^2 ds + 4\alpha t^\alpha \int_0^t (t-s)^{\alpha-1} E|H^k(s) - H(s)|^2 ds \leq \\ &4M^2 E|e^{k+1}(t-\tau)|^2 + 4tL_1\int_0^t E(|e^{k+1}(s)|^2 + |e^k(s)|^2 + |e^{k+1}(s-\tau)|^2 + |e^k(s-\tau)|^2)ds + \\ &4L_2\int_0^t E(|e^{k+1}(s)|^2 + |e^k(s)|^2 + |e^{k+1}(s-\tau)|^2 + |e^k(s-\tau)|^2)ds + \\ &4\alpha t^\alpha L_3\int_0^t (t-s)^{\alpha-1} E(|e^{k+1}(s)|^2 + |e^k(s)|^2 + |e^{k+1}(s-\tau)|^2 + |e^k(s-\tau)|^2)ds. \end{aligned}$$

再由式 (5) 可得  $E|e^{k+1}(t)| \leq 4(tL_1 + L_2)\int_0^t E(|e^{k+1}(s)|^2 + |e^k(s)|^2)ds + 6\alpha t^\alpha L_3\int_0^t (t-s)^{\alpha-1} E(|e^{k+1}(s)|^2 + |e^k(s)|^2)ds$ . 令  $\epsilon^{k+1}(t) = E|e^{k+1}(t)|^2$ ,  $k=0,1,2,\dots$ , 则有:

$$\epsilon^{k+1}(t) \leq \hat{a}_1\int_0^t (t-s)^{\alpha-1}\epsilon^{k+1}(s)ds + \hat{b}_1\int_0^t \epsilon^{k+1}(s)ds + \hat{a}_1\int_0^t (t-s)^{\alpha-1}\epsilon^k(s)ds + \hat{b}_1\int_0^t \epsilon^k(s)ds,$$

其中  $\hat{a}_1 = 4\alpha t^\alpha L_3$ ,  $\hat{b}_1 = 4(tL_1 + L_2)$ .

再根据算子  $\mathcal{T}_1, \mathcal{T}_2$  的定义和可交换性以及引理 3 得:

$$\begin{aligned} \epsilon^{k+1}(t) &\leq ((\hat{a}_1\mathcal{T}_2 + \hat{b}_1\mathcal{T}_1)\epsilon^k)(t) + \sum_{n=1}^{\infty} \sum_{i=0}^n \binom{n}{i} (\hat{b}_1)^{n-i} (\hat{a}_1)^i \left( \frac{(\Gamma(\alpha))^i}{\Gamma(i\alpha + n - i)} \right) \times \\ &\int_0^t (t-s)^{i\alpha - (i+1-n)} ((\hat{a}_1\mathcal{T}_2 + \hat{b}_1\mathcal{T}_1)\epsilon^k(s))ds. \end{aligned} \quad (6)$$

由于

$$\begin{aligned} (\mathcal{T}_2^i \mathcal{T}_1^{n-i} \varphi)(t) &= \frac{(\Gamma(\alpha))^i}{\Gamma(n-i)\Gamma(i\alpha)} \int_0^t \int_0^s (s-\delta)^{i\alpha + n - i - 2} \varphi(\delta) d\delta ds = \\ &\frac{(\Gamma(\alpha))^i}{\Gamma(n-i)\Gamma(i\alpha)} \int_0^t \varphi(\delta) \int_\delta^t (s-\delta)^{i\alpha + n - i - 2} ds d\delta = \\ &\frac{(\Gamma(\alpha))^i}{(i\alpha + n - i - 1)\Gamma(n-i)\Gamma(i\alpha)} \int_0^t (t-\delta)^{i\alpha + n - i - 1} \varphi(\delta) d\delta, \end{aligned}$$

因此

$$\int_0^t (t-\delta)^{i\alpha + n - i - 1} \varphi(\delta) d\delta = \frac{(i\alpha + n - i - 1)\Gamma(n-i)\Gamma(i\alpha)}{\Gamma^i(\alpha)} (\mathcal{T}_2^i \mathcal{T}_1^{n-i} \varphi)(t). \quad (7)$$

根据算子  $\mathcal{T}_1, \mathcal{T}_2$  的可交换性, 将式 (7) 带入式 (6) 可得:

$$\begin{aligned} \epsilon^{k+1}(t) &\leq ((\hat{a}_1\mathcal{T}_2 + \hat{b}_1\mathcal{T}_1)\epsilon^k)(t) + \sum_{n=1}^{\infty} \sum_{i=0}^n \binom{n}{i} (\hat{b}_1)^{n-i} (\hat{a}_1)^i \frac{\Gamma(n-i)\Gamma(i\alpha)}{\Gamma(i\alpha + n - i - 1)} \times \\ &(\mathcal{T}_2^i \mathcal{T}_1^{n-i} (\hat{a}_1\mathcal{T}_2 + \hat{b}_1\mathcal{T}_1)\epsilon^k)(t) \leq ((\hat{a}_1\mathcal{T}_2 + \hat{b}_1\mathcal{T}_1)\epsilon^k)(t) + \end{aligned}$$

$$\sum_{n=1}^{\infty} \sum_{i=0}^n \binom{n}{i} (\hat{b}_1)^{n-i} (\hat{a}_1)^i \frac{\Gamma(n-i)\Gamma(i\alpha)}{\Gamma(i\alpha+n-i-1)} (\hat{a}_1 \mathcal{T}_1^{n-i} \mathcal{T}_2^{i+1} + \hat{b}_1 \mathcal{T}_1^{n-i+1} \mathcal{T}_2^i) \epsilon^k(t).$$

由文献[7]中的性质 2.2 知,算子  $\mathcal{T}_1, \mathcal{T}_2$  对  $\varphi(t) \in C(\Omega, \mathbf{R}^n)$  是非递减的. 对  $k$  进行数学归纳可得:

$$((\hat{a}_1 \mathcal{T}_2 + \hat{b}_1 \mathcal{T}_1) \epsilon^{k+1})(t) \leq ((\hat{a}_1 \mathcal{T}_2 + \hat{b}_1 \mathcal{T}_1)^{k+1} \epsilon^0)(t). \quad (8)$$

式(8)中:当  $k=1$  时,显然知  $((\hat{a}_1 \mathcal{T}_2 + \hat{b}_1 \mathcal{T}_1) \epsilon^2)(t) \leq ((\hat{a}_1 \mathcal{T}_2 + \hat{b}_1 \mathcal{T}_1)^1 \epsilon^1)(t)$  成立. 假设式(8)对任意  $k(k=1, 2, \dots)$  成立, 则需证明  $k+1$  时式(8)仍然成立, 于是有  $((\hat{a}_1 \mathcal{T}_2 + \hat{b}_1 \mathcal{T}_1) \epsilon^{k+1})(t) \leq ((\hat{a}_1 \mathcal{T}_2 + \hat{b}_1 \mathcal{T}_1)(\hat{a}_1 \mathcal{T}_2 + \hat{b}_1 \mathcal{T}_1)^k \epsilon^0)(t) \leq ((\hat{a}_1 \mathcal{T}_2 + \hat{b}_1 \mathcal{T}_1)^{k+1} \epsilon^0)(t)$ . 因此有

$$\epsilon^{k+1}(t) \leq ((\hat{a}_1 \mathcal{T}_2 + \hat{b}_1 \mathcal{T}_1)^{k+1} \epsilon^0)(t) + \sum_{n=1}^{\infty} \sum_{i=0}^n \binom{n}{i} (\hat{b}_1)^{n-i} (\hat{a}_1)^i \frac{\Gamma(n-i)\Gamma(i\alpha)}{\Gamma(i\alpha+n-i-1)} \times (\hat{a}_1 \mathcal{T}_1^{n-i} \mathcal{T}_2^{i+1} + \hat{b}_1 \mathcal{T}_1^{n-i+1} \mathcal{T}_2^i)^{k+1} \epsilon^0(t).$$

由引理 1 知算子  $\mathcal{T}_1, \mathcal{T}_2$  可交换. 再由引理 4 可知  $\mathcal{T}_1, \mathcal{T}_2$  是定义在  $C(\Omega, \mathbf{R}^n)$  上的紧算子, 且  $\sigma(\mathcal{T}_1) = \sigma(\mathcal{T}_2) = \{0\}$ , 因此  $\{x^{k+1}(t)\}$  在区间  $[0, \tau]$  上均方一致收敛, 即

$$\lim_{k \rightarrow \infty} E(\sup_{0 \leq t \leq \tau} |e^{k+1}(t)|^2) = 0. \quad (9)$$

当  $t \in [\tau, 2\tau]$  时, 由式(4)减去式(3)可得:

$$x^{k+1}(t) - x(t) = x^{k+1}(\tau) - x(\tau) + D(x^{k+1}(t - \tau)) - D(x(t - \tau)) + D(x^{k+1}(\tau)) - D(x(\tau)) + \int_{\tau}^t (F^k(s) - F(s)) ds + \int_{\tau}^t (G^k(s) - G(s)) d\mathbf{B}(s) + \alpha \int_{\tau}^t (t-s)^{\alpha-1} (H^k(s) - H(s)) ds.$$

于是利用  $|a+b+c+d+e+f|^2 \leq 6|a|^2 + 6|b|^2 + 6|c|^2 + 6|d|^2 + 6|e|^2 + 6|f|^2$  和 Hölder 不等式可得:

$$\begin{aligned} |e^{k+1}(t)|^2 &\leq 6|x^{k+1}(\tau) - x(\tau)|^2 + 6|D(x^{k+1}(t - \tau)) - D(x(t - \tau))|^2 + \\ &6|D(x^{k+1}(\tau)) - D(x(\tau))|^2 + 6\left|\int_{\tau}^t (F^k(s) - F(s)) ds\right|^2 + 6\left|\int_{\tau}^t (G^k(s) - G(s)) d\mathbf{B}(s)\right|^2 + \\ &6\left|\alpha \int_{\tau}^t (t-s)^{\alpha-1} (H^k(s) - H(s)) ds\right|^2. \end{aligned}$$

对上式两边同时取期望, 并利用假设 1、假设 2 和 Itô 积分的等距性可得:

$$\begin{aligned} E|e^{k+1}(t)|^2 &\leq 6(M^2 + 1)E|e^{k+1}(\tau)|^2 + 6M^2 E|e^{k+1}(t - \tau)|^2 + \\ &6[(t - \tau)L_1 + L_2] \int_{\tau}^t E(|e^{k+1}(s)|^2 + |e^k(s)|^2 + |e^{k+1}(s - \tau)|^2 + |e^k(s - \tau)|^2) ds + \\ &6\alpha(t - \tau)^{\alpha} L_3 \int_{\tau}^t (t-s)^{\alpha-1} E(|e^{k+1}(s)|^2 + |e^k(s)|^2 + |e^{k+1}(s - \tau)|^2 + |e^k(s - \tau)|^2) ds. \end{aligned}$$

令  $\epsilon^{k+1}(t) = E|e^{k+1}(t)|^2$ ,  $k=0, 1, 2, \dots$ . 于是当  $t \in [\tau, 2\tau]$  时根据式(9)可得:

$$\epsilon^{k+1}(t) \leq \hat{a}_2 \int_{\tau}^t (t-s)^{\alpha-1} \epsilon^{k+1}(s) ds + \hat{b}_2 \int_{\tau}^t \epsilon^{k+1}(s) ds + \hat{a}_2 \int_{\tau}^t (t-s)^{\alpha-1} \epsilon^k(s) ds + \hat{b}_2 \int_{\tau}^t \epsilon^k(s) ds,$$

其中  $\hat{a}_2 = 6\alpha(t - \tau)^{\alpha} L_3$ ,  $\hat{b}_2 = 6[(t - \tau)L_1 + L_2]$ .

按上述归纳法处理算子  $\mathcal{T}_1$  和  $\mathcal{T}_2$  可得:

$$\begin{aligned} \epsilon^{k+1}(t) &\leq ((\hat{a}_2 \mathcal{T}_2 + \hat{b}_2 \mathcal{T}_1) \epsilon^k)(t) + \sum_{n=1}^{\infty} \sum_{i=0}^n \binom{n}{i} (\hat{b}_2)^{n-i} (\hat{a}_2)^i \left( \frac{\Gamma(\alpha)}{\Gamma(i\alpha + n - i)} \right) \times \\ &\int_0^t (t-s)^{i\alpha - (i+1-n)} ((\hat{a}_2 \mathcal{T}_2 + \hat{b}_2 \mathcal{T}_1) \epsilon^k(s)) ds \leq ((\hat{a}_2 \mathcal{T}_2 + \hat{b}_2 \mathcal{T}_1)^{k+1} \epsilon^0)(t) + \\ &\sum_{n=1}^{\infty} \sum_{i=0}^n \binom{n}{i} (\hat{b}_2)^{n-i} (\hat{a}_2)^i \frac{\Gamma(n-i)\Gamma(i\alpha)}{\Gamma(i\alpha+n-i-1)} (\hat{a}_2 \mathcal{T}_1^{n-i} \mathcal{T}_2^{i+1} + \hat{b}_2 \mathcal{T}_1^{n-i+1} \mathcal{T}_2^i)^{k+1} \epsilon^0(s). \end{aligned}$$

根据算子  $\mathcal{T}_1, \mathcal{T}_2$  可交换及引理 4 知,  $\mathcal{T}_1, \mathcal{T}_2$  是定义在  $C(\Omega, \mathbf{R}^n)$  上的紧算子, 且  $\sigma(\mathcal{T}_1) = \sigma(\mathcal{T}_2) = \{0\}$ . 由

此可知,  $\{x^{k+1}(t)\}$  在区间  $[\tau, 2\tau]$  上均方一致收敛, 即

$$\lim_{k \rightarrow \infty} E \left( \sup_{\tau \leq t \leq 2\tau} |e^{k+1}(t)|^2 \right) = 0.$$

当  $t \in [n\tau, (n+1)\tau]$ ,  $n=0, 1, \dots$  时, 重复以上过程可得

$$\lim_{k \rightarrow \infty} E \left( \sup_{n\tau \leq t \leq (n+1)\tau} |e^{k+1}(t)|^2 \right) = 0.$$

定理 1 证毕.

### 3 数值模拟

由于难以求出分数阶中立型随机时滞微分方程的解析解, 因此本文将分数阶中立型随机时滞微分方程的隐式 Euler-Maruyama 数值解作为真解. 设  $0 < \alpha \leq 1$ , 并考虑如下形式的初值问题:

$$\begin{cases} d[x(t) - D(x(t))] = f(x(t))dt + g(x(t))d\mathbf{B}(t) + \Gamma(\alpha + 1)dI_{0+}^{\alpha}(\sigma(x(t))), & t \in [0, T]; \\ x(0) = x_0, & t \in [0, T]. \end{cases} \quad (10)$$

下面用 Euler-Maruyama 方法给出方程(10)的数值解. 首先将区间  $[0, T]$  分成  $M$  个子区间  $[t_{m-1}, t_m]$ ,

其中  $m=1, 2, \dots, L > 0$ ,  $\Delta t = \frac{T}{L}$ ,  $t_m = m\Delta t$ , 则式(10)真解的表达式为:

$$\begin{aligned} x(t_m) &= x(t_{m-1}) + D(x(t_m)) - D(x(t_{m-1})) + \int_{t_{m-1}}^{t_m} f(x(s))ds + \int_{t_{m-1}}^{t_m} g(x(s))d\mathbf{B}(s) + \\ &\quad \alpha \int_0^{t_m} (t_m - s)^{\alpha-1} \sigma(x(s))ds - \alpha \int_0^{t_{m-1}} (t_m - s)^{\alpha-1} \sigma(x(s))ds = \\ &\quad x(t_{m-1}) + D(x(t_m)) - D(x(t_{m-1})) + \int_{t_{m-1}}^{t_m} f(x(s))ds + \int_{t_{m-1}}^{t_m} g(x(s))d\mathbf{B}(s) + \\ &\quad \alpha \int_0^{t_{m-1}} (t_m - s)^{\alpha-1} \sigma(x(s))ds + \alpha \int_{t_{m-1}}^{t_m} (t_m - s)^{\alpha-1} \sigma(x(s))ds - \alpha \int_0^{t_{m-1}} (t_m - s)^{\alpha-1} \sigma(x(s))ds = \\ &\quad I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8. \\ I_6 &= \alpha \int_0^{t_{m-1}} (t_m - s)^{\alpha-1} \sigma(x(s))ds = \alpha \sum_{k=1}^{m-1} \int_{(k-1)\Delta t}^{k\Delta t} (t_m - s)^{\alpha-1} \sigma(x(s))ds = \\ &\quad (\Delta t)^{\alpha} \sum_{k=1}^{m-1} [(m-k+1)^{\alpha} - (m-k)^{\alpha}] \int_{(k-1)\Delta t}^{k\Delta t} \sigma(x(s))ds + o(\Delta t) = \lambda \sum_{k=1}^{m-1} \omega_{m-k+1}^{\alpha} \sigma(x_k(t)) + o(\Delta t). \end{aligned}$$

其中  $\lambda = (\Delta t)^{\alpha}$ ,  $\omega_{m-k+1}^{\alpha} = (m-k+1)^{\alpha} - (m-k)^{\alpha}$ ,  $\sigma(x_k(t)) = \int_{(k-1)\Delta t}^{k\Delta t} \sigma(x(s))ds$ . 同理可得:

$$I_7 = \alpha \int_{t_{m-1}}^{t_m} (t_m - s)^{\alpha-1} \sigma(x(s))ds = \lambda \sigma(x_m(t)) + o(\Delta t),$$

$$I_8 = \alpha \int_0^{t_{m-1}} (t_{m-1} - s)^{\alpha-1} \sigma(x(s))ds = \lambda \sum_{k=1}^{m-1} \omega_{m-k}^{\alpha} \sigma(x_k(t)) + o(\Delta t).$$

由此可得方程(10)的数值解为:

$$\begin{aligned} x_m &= x_{m-1} + D(x_m) - D(x_{m-1}) + f(x_{m-1})\Delta t + g(x_{m-1})(\mathbf{B}(t_m) - \mathbf{B}(t_{m-1})) + \\ &\quad \lambda \sum_{k=1}^{m-1} (\omega_{m-k+1}^{\alpha} - \omega_{m-k}^{\alpha}) \sigma(x_k(t)) + \lambda \sigma(x_m(t)). \end{aligned}$$

**例 1** 考虑如下分数阶中立型随机时滞微分方程的初值问题:

$$\begin{cases} d \left[ x(t) - \frac{1}{4}x(t-1) \right] = (-x(t) + 6x(t-1))dt + x(t)d\mathbf{B}(t) + \\ \quad \Gamma(\alpha + 1)dI_{0+}^{\alpha}(x(t)), & t \in [0, T]; \\ x(t) = t + 1, & t \in [-1, 0]. \end{cases} \quad (11)$$

首先用波形松弛方法和 Euler-Maruyama 方法得出式(11)的解,然后将波形松弛解与真解进行比较和分析.用 Matlab 软件(取  $\alpha = 0.7$ ,  $T = 1$ ,  $M = 2^{10}$ ) 模拟出的式(11)的波形松弛解如图 1 所示,用 Mathab 软件模拟出的式(11)的波形松弛解和真解之间的误差如图 2 所示.由图 1 和图 2 可以看出,用波形松弛方法求解分数阶中立型随机时滞微分方程是有效的,且波形松弛方法在均方意义上是收敛的.

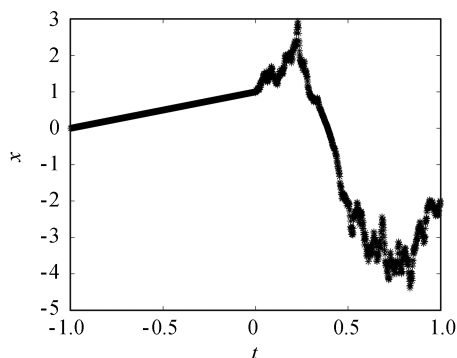


图 1  $\alpha=0.7$  时式(11)的波形松弛解

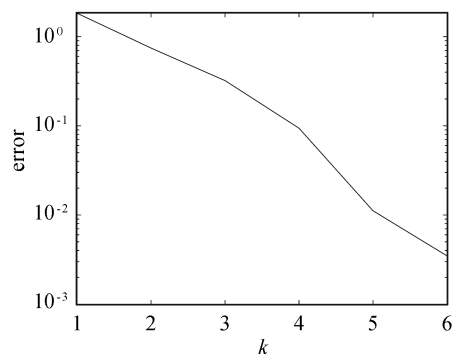


图 2  $\alpha=0.7$  时式(11)在迭代次数  $k$  下的波形松弛解和利用隐式 Euler-Maruyama 法求得的数值解的误差

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