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一类具反馈控制的偏利模型平衡点的稳定性

吴丽萍
(闽江学院 数学与数据科学学院(软件学院)，福州 350108)

摘要：研究一类具有反馈控制变量和出生率具有密度制约的偏利共生模型的平衡点稳定性. 首先通过分析系统的平衡点，得到系统正平衡点和边界平衡点存在的条件；其次通过构造适当的 Lyapunov 函数，得到在一定条件下系统的正平衡点和边界平衡点是全局稳定的；最后利用数值模拟验证了所得结果的正确性.
关键词：偏利模型；反馈控制；出生率密度制约；全局稳定
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Stability of the equilibrium for a commensal symbiosis model with feedback controls

WU Liping
(*College of Mathematics and Data Science (Software College), Minjiang University, Fuzhou* 350108, *China*)

Abstract: Stability of the equilibrium for a commensal symbiosis model with density dependent birth rate and feedback controls is investigated. Firstly, the equilibrium of the system is analyzed, and the conditions for the existence of positive equilibrium and the boundary equilibrium are obtained. Secondly, by constructing some suitable Lyapunov functions, the global stability of the positive equilibrium and the boundary equilibrium is showed under some suitable assumptions. Finally, the conclusions are verified by data simulation.
Keywords: commensal symbiosis model; feedback controls; density dependent birth rate; global stability

0　引言

近年来，许多学者对两种群之间偏利共生作用的模型进行了研究，并取得了一些良好的研究结果[1-4]；但这些文献所研究的模型都是基于传统的 Logistic 模型，且模型中的种群出生率都为常数(不受种群的密度制约). 研究[5]显示，在某些情形下建立出生率具有密度制约的种群更为合理. 2018 年，Chen[6]提出了如下出生率具有密度制约的偏利共生模型：

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t}=x\left(\dfrac{b_{11}}{b_{12}+b_{13}x}-b_{14}-a_{11}x+a_{12}y\right), \\ \dfrac{\mathrm{d}y}{\mathrm{d}t}=y\left(\dfrac{b_{21}}{b_{22}+b_{23}y}-b_{24}-a_{22}y\right). \end{cases} \tag{1}$$

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作者简介：吴丽萍(1972—)，女，硕士，副教授，研究方向为生物数学.

其中：$\dfrac{b_{11}}{b_{12}+b_{13}x}$ 和 $\dfrac{b_{21}}{b_{22}+b_{23}y}$ 分别表示第 1 个种群和第 2 个种群的出生率，且具有密度制约；b_{ij}($i=1,2$；$j=1,2,3,4$)、a_{11}、a_{12} 和 a_{22} 均为正数，其生物学含义见文献[6]. 文献[6] 的作者通过构造适当的 Lyapunov 函数，在一定条件下得到系统的所有平衡点是全局渐近稳定的. 考虑到系统在实际的生态环境中会受到不定因素的干扰，因此本文研究如下具有反馈控制变量和出生率具有密度制约的偏利共生模型：

$$\begin{cases}\dfrac{\mathrm{d}x}{\mathrm{d}t}=x\left(\dfrac{b_{11}}{b_{12}+b_{13}x}-b_{14}-a_{11}x+a_{12}y-\alpha_1u_1\right),\\ \dfrac{\mathrm{d}y}{\mathrm{d}t}=y\left(\dfrac{b_{21}}{b_{22}+b_{23}y}-b_{24}-a_{22}y-\alpha_2u_2\right),\\ \dfrac{\mathrm{d}u_1}{\mathrm{d}t}=-\eta_1u_1+a_1x,\\ \dfrac{\mathrm{d}u_2}{\mathrm{d}t}=-\eta_2u_2+a_2y.\end{cases}\tag{2}$$

其中：系统中所有参数均为正数，$x(t)$ 和 $y(t)$ 分别是两个种群在 t 时刻的密度，u_i($i=1,2$) 是控制变量.

1　平衡点的存在性

首先考虑系统(2) 的正平衡点. 系统(2) 的正平衡点(x^*,y^*,u_1^*,u_2^*) 满足如下方程组：

$$\begin{cases}\dfrac{b_{11}}{b_{12}+b_{13}x}-b_{14}-a_{11}x+a_{12}y-\alpha_1u_1=0,\\ \dfrac{b_{21}}{b_{22}+b_{23}y}-b_{24}-a_{22}y-\alpha_2u_2=0,\\ -\eta_1u_1+a_1x=0,\\ -\eta_2u_2+a_2y=0.\end{cases}\tag{3}$$

由 $-\eta_1u_1+a_1x=0$，$-\eta_2u_2+a_2y=0$ 分别可得 $u_1^*=\dfrac{a_1}{\eta_1}x^*$，$u_2^*=\dfrac{a_2}{\eta_2}y^*$. 记 $\Delta_1=(\eta_2b_{23}b_{24}+\eta_2a_{22}b_{22}+\alpha_2a_2b_{22})^2-4\eta_2b_{23}(\eta_2a_{22}+\alpha_2a_2)(b_{22}b_{24}-b_{21})$，则当 $b_{22}b_{24}-b_{21}<0$，即 $b_{21}>b_{22}b_{24}$ 时，方程 $\dfrac{b_{21}}{b_{22}+b_{23}y}-b_{24}-a_{22}y-\dfrac{\alpha_2a_2}{\eta_2}y=0$ 有唯一的正解 $y^*=\dfrac{-(\eta_2b_{23}b_{24}+\eta_2a_{22}b_{22}+\alpha_2a_2b_{22})+\sqrt{\Delta_1}}{2b_{23}(\eta_2a_{22}+\alpha_2a_2)}$. 记 $\Delta_2=(\eta_1b_{13}b_{14}+\eta_1a_{11}b_{12}+\alpha_1a_1b_{12}-\eta_1a_{12}b_{13}y^*)^2-4\eta_1b_{13}(\eta_1a_{11}+\alpha_1a_1)(b_{12}b_{14}-b_{11}-a_{12}b_{12}y^*)$，则当 $b_{14}b_{12}-b_{11}-a_{12}b_{12}y^*<0$，即 $\dfrac{b_{11}}{b_{12}}+a_{12}y^*>b_{14}$ 时，方程 $\dfrac{b_{11}}{b_{12}+b_{13}x}-b_{14}-a_{11}x+a_{12}y^*-\dfrac{\alpha_1a_1}{\eta_1}x=0$ 有唯一的正解 $x^*=\dfrac{-(\eta_1b_{13}b_{14}+\eta_1a_{11}b_{12}+\alpha_1a_1b_{12}-\eta_1a_{12}b_{13}y^*)+\sqrt{\Delta_2}}{2b_{13}(\eta_1a_{11}+\alpha_1a_1)}$. 注意到 $\dfrac{b_{11}}{b_{12}}>b_{14}$ 成立时 $\dfrac{b_{11}}{b_{12}}+a_{12}y^*>b_{14}$ 也成立，于是可得如下引理 1.

引理 1　如果 $b_{11}>b_{12}b_{14}$，$b_{21}>b_{22}b_{24}$，则系统(2) 有唯一的正平衡点(x^*,y^*,u_1^*,u_2^*).

下面考虑系统(2) 的边界平衡点. 易知点(0,0,0,0) 是系统(2) 的一个边界平衡点. 记 $\Delta_3=(\eta_1b_{13}b_{14}+\eta_1a_{11}b_{12}+\alpha_1a_1b_{12})^2-4\eta_1b_{13}(\eta_1a_{11}+\alpha_1a_1)(b_{12}b_{14}-b_{11})$，于是类似于上面的讨论可得以下引理：

引理 2　如果 $b_{11}>b_{12}b_{14}$，则系统(2) 有边界平衡点($x^{**},0,u_1^{**},0$)，其中

$$x^{**}=\frac{-(\eta_1b_{13}b_{14}+\eta_1a_{11}b_{12}+\alpha_1a_1b_{12})+\sqrt{\Delta_3}}{2b_{13}(\eta_1a_{11}+\alpha_1a_1)},\quad u_1^{**}=\frac{a_1}{\eta_1}x^{**}.$$

引理 3　如果 $b_{21}>b_{22}b_{24}$，则系统(2) 有边界平衡点($0,y^*,0,u_2^*$).

2 平衡点的稳定性

定理1 设(x,y,u_1,u_2)是系统(2)的任意正解. 若$b_{11}>b_{12}b_{14}$, $b_{21}>b_{22}b_{24}$, 则系统(2)的正平衡点(x^*,y^*,u_1^*,u_2^*)是全局稳定的.

证明 构造Lyapunov函数:

$$V_1(t)=\delta_1\left(x-x^*-x^*\ln\frac{x}{x^*}\right)+\delta_2\left(y-y^*-y^*\ln\frac{y}{y^*}\right)+\delta_3(u_1-u_1^*)^2+\delta_4(u_2-u_2^*)^2,$$

其中$\delta_i(i=1,2,3,4)$是待定的正常数. 注意到系统(2)的正平衡点(x^*,y^*,u_1^*,u_2^*)满足方程组(3), 因此沿着系统(2)的正解计算$V(t)$的右上导数可得:

$$\begin{aligned}D^+V_1(t)=&\delta_1(x-x^*)\left(\frac{b_{11}}{b_{12}+b_{13}x}-b_{14}-a_{11}x+a_{12}y-\alpha_1u_1\right)+\\&\delta_2(y-y^*)\left(\frac{b_{21}}{b_{22}+b_{23}y}-b_{24}-a_{22}y-\alpha_2u_2\right)+\\&2\delta_3(u_1-u_1^*)(-\eta_1u_1+a_1x)+2\delta_4(u_2-u_2^*)(-\eta_2u_2+a_2y)=\\&\delta_1(x-x^*)\left(\frac{b_{11}}{b_{12}+b_{13}x}-\frac{b_{11}}{b_{12}+b_{13}x^*}+a_{11}x^*-a_{12}y^*+\alpha_1u_1^*-a_{11}x+a_{12}y-\alpha_1u_1\right)+\\&\delta_2(y-y^*)\left(\frac{b_{21}}{b_{22}+b_{23}y}-\frac{b_{21}}{b_{22}+b_{23}y^*}+a_{22}y^*+\alpha_2u_2^*-a_{22}y-\alpha_2u_2\right)+2\delta_3(u_1-u_1^*)\times\\&(-\eta_1u_1+a_1x+\eta_1u_1^*-a_1x^*)+2\delta_4(u_2-u_2^*)(-\eta_2u_2+a_2y+\eta_2u_2^*-a_2y^*)=\\&-\frac{\delta_1b_{11}b_{13}}{(b_{12}+b_{13}x)(b_{12}+b_{13}x^*)}(x-x^*)^2-\\&[\delta_1a_{11}(x-x^*)^2-\delta_1a_{12}(x-x^*)(y-y^*)+\delta_2a_{22}(y-y^*)^2]-\\&\frac{\delta_2b_{21}b_{23}}{(b_{22}+b_{23}y)(b_{22}+b_{23}y^*)}(y-y^*)^2-2\delta_3\eta_1(u_1-u_1^*)^2-2\delta_4\eta_2(u_2-u_2^*)^2+\\&(2\delta_3a_1-\delta_1\alpha_1)(x-x^*)(u_1-u_1^*)+(2\delta_4a_2-\delta_2\alpha_2)(u_2-u_2^*)(y-y^*).\end{aligned}$$

在上式中取$\delta_1=\frac{a_{11}a_{22}}{a_{12}^2}$, $\delta_2=\frac{1}{4}$, $\delta_3=\frac{\delta_1\alpha_1}{2a_1}$, $\delta_4=\frac{\delta_2\alpha_2}{2a_2}$, 则有:

$$\begin{aligned}D^+V_1(t)=&-\frac{\delta_1b_{11}b_{13}}{(b_{12}+b_{13}x)(b_{12}+b_{13}x^*)}(x-x^*)^2-[\sqrt{\delta_1a_{11}}(x-x^*)-\sqrt{\delta_2a_{22}}(y-y^*)]^2-\\&\frac{\delta_2b_{21}b_{23}}{(b_{22}+b_{23}y)(b_{22}+b_{23}y^*)}(y-y^*)^2-2\delta_3\eta_1(u_1-u_1^*)^2-2\delta_4\eta_2(u_2-u_2^*)^2.\end{aligned}$$

上式表明$\frac{\mathrm{d}V_1}{\mathrm{d}t}\leqslant 0$, 当且仅当$x=x^*$, $y=y^*$, $u_1=u_1^*$, $u_2=u_2^*$时取等号. 于是由Lyapunov稳定性定理[7]可知, 系统(2)的正平衡点(x^*,y^*,u_1^*,u_2^*)是全局稳定的. 证毕.

定理2 设(x,y,u_1,u_2)是系统(2)的任意正解. 若$b_{11}>b_{12}b_{14}$, $b_{22}b_{24}>b_{21}$, 则系统(2)的边界平衡点$(x^{**},0,u_1^{**},0)$是全局稳定的.

证明 构造Lyapunov函数:

$$V_2(t)=\beta_1\left(x-x^{**}-x^{**}\ln\frac{x}{x^{**}}\right)+\beta_2y+\beta_3(u_1-u_1^{**})^2+\beta_4u_2^2,$$

其中$\beta_i(i=1,2,3,4)$是待定的正常数. 注意到系统(2)的边界平衡点$(x^{**},0,u_1^{**},0)$满足$\frac{b_{11}}{b_{12}+b_{13}x^{**}}-b_{14}-a_{11}x^{**}-\alpha_1u_1^{**}=0$, 因此有:

$$D^+V_2(t)=\beta_1(x-x^{**})(\frac{b_{11}}{b_{12}+b_{13}x}-b_{14}-a_{11}x+a_{12}y-\alpha_1u_1)+$$
$$\beta_2y(\frac{b_{21}}{b_{22}+b_{23}y}-b_{24}-a_{22}y-\alpha_2u_2)+2\beta_3(u_1-u_1^{**})(-\eta_1u_1+a_1x)+$$
$$2\beta_4u_2(-\eta_2u_2+a_2y)=\beta_1(x-x^{**})(\frac{b_{11}}{b_{12}+b_{13}x}-\frac{b_{11}}{b_{12}+b_{13}x^{**}}+a_{11}x^{**}+$$
$$\alpha_1u_1^{**}-a_{11}x+a_{12}y-\alpha_1u_1)+\beta_2y(\frac{b_{21}}{b_{22}+b_{23}y}-b_{24})-\beta_2a_{22}y^2-\beta_2\alpha_2u_2y+$$
$$2\beta_3(u_1-u_1^{**})(-\eta_1u_1+a_1x+\eta_1u_1^{**}-a_1x^{**})-2\beta_4\eta_2u_2^2+2\beta_4a_2u_2y\leqslant$$
$$-\frac{\beta_1b_{11}b_{13}}{(b_{12}+b_{13}x)(b_{12}+b_{13}x^{**})}(x-x^{**})^2-[\beta_1a_{11}(x-x^{**})^2-$$
$$\beta_1a_{12}y(x-x^{**})+\beta_2a_{22}y^2]+\beta_2(\frac{b_{21}}{b_{22}}-b_{24})y-2\beta_3\eta_1(u_1-u_1^{**})^2-$$
$$2\beta_4\eta_2u_2^2+(2\beta_3a_1-\beta_1\alpha_1)(x-x^{**})(u_1-u_1^{**})+(2\beta_4a_2-\beta_2\alpha_2)u_2y.$$

在上式中取 $\beta_1=\frac{a_{11}a_{22}}{a_{12}^2}$，$\beta_2=\frac{1}{4}$，$\beta_3=\frac{\beta_1\alpha_1}{2a_1}$，$\beta_4=\frac{\beta_2\alpha_2}{2a_2}$，则有：

$$D^+V_2(t)=-\frac{\beta_1b_{11}b_{13}}{(b_{12}+b_{13}x)(b_{12}+b_{13}x^{**})}(x-x^{**})^2-[\sqrt{\beta_1a_{11}}(x-x^{**})-\sqrt{\beta_2a_{22}}\,y]^2-$$
$$\beta_2(b_{24}-\frac{b_{21}}{b_{22}})y^2-2\beta_3\eta_1(u_1-u_1^{**})^2-2\beta_4\eta_2u_2^2\leqslant 0,$$

当且仅当 $x=x^{**}$，$y=0$，$u_1=u_1^{**}$，$u_2=0$ 时取等号. 由 Lyapunov 稳定性定理[7] 可知，系统(2) 的边界平衡点$(x^{**},0,u_1^{**},0)$ 是全局稳定的. 证毕.

引理 4[6]　如果 $a>bd$，则方程 $\frac{\mathrm{d}y}{\mathrm{d}t}=y(\frac{a}{b+cy}-d-ey)$ 的正平衡点 $\bar{y}$ 是全局稳定的.

定理 3　设(x,y,u_1,u_2)是系统(2) 的任意正解. 若 $b_{21}>b_{22}b_{24}$ 且 $\frac{b_{11}}{b_{12}}+a_{12}y^*<b_{14}$，则系统(2) 的边界平衡点$(0,y^*,0,u_2^*)$ 是全局稳定的.

证明　由 $\frac{b_{11}}{b_{12}}+a_{12}y^*<b_{14}$ 知，存在充分小的 $\varepsilon>0$，使得 $\frac{b_{11}}{b_{12}}+a_{12}(y^*+\varepsilon)<b_{14}$. 注意到 y^* 是方程 $\frac{\mathrm{d}y}{\mathrm{d}t}=y[\frac{b_{21}}{b_{22}+b_{23}y}-b_{24}-(a_{22}+\frac{\alpha_2}{\eta_2a_2})y]$ 的正平衡点，因此由引理 4 知，$\lim\limits_{t\to+\infty}y(t)=y^*$. 所以对上述 $\varepsilon>0$ 存在 T_1，且当 $t>T_1$ 时有 $y(t)<y^*+\varepsilon$. 构造 Lyapunov 函数：

$$V_3(t)=x+\gamma_1(y-y^*-y^*\ln\frac{y}{y^*})+\gamma_2u_1^2+\gamma_3(u_2-u_2^*)^2,$$

其中 $\gamma_i(i=1,2,3)$ 是待定的正常数. 因为系统(2) 的边界平衡点$(0,y^*,0,u_2^*)$ 满足$\frac{b_{21}}{b_{22}+b_{23}y^*}-b_{24}-a_{22}y^*-\alpha_2u_2^*=0$，所以当 $t>T_1$ 时有

$$D^+V_3(t)=x(\frac{b_{11}}{b_{12}+b_{13}x}-b_{14}-a_{11}x+a_{12}y-\alpha_1u_1)+2\gamma_2u_1(-\eta_1u_1+a_1x)+$$
$$\gamma_1(y-y^*)(\frac{b_{21}}{b_{22}+b_{23}y}-b_{24}-a_{22}y-\alpha_2u_2)+2\gamma_3(u_2-u_2^*)(-\eta_2u_2+a_2y)\leqslant$$
$$(\frac{b_{11}}{b_{12}}-b_{14})x-a_{11}x^2+a_{12}(y^*+\varepsilon)x-\alpha_1xu_1-2\gamma_2\eta_1u_1^2+2\gamma_2a_1xu_1+$$

$$\gamma_1(y-y^*)(\frac{b_{21}}{b_{22}+b_{23}y}-\frac{b_{21}}{b_{22}+b_{23}y}+a_{22}y^*+\alpha_2u_2^*-a_{22}y-\alpha_2u_2)+$$
$$2\gamma_3(u_2-u_2^*)(-\eta_2u_2+a_2y+\eta_2u_2^*-a_2y^*)\leqslant$$
$$[\frac{b_{11}}{b_{12}}+a_{12}(y^*+\varepsilon)-b_{14}]x-a_{11}x^2+(2\gamma_2a_1-\alpha_1)xu_1-2\gamma_2\eta_1u_1^2-2\gamma_3\eta_2(u_2-u_2^*)^2-$$
$$[\frac{\gamma_1b_{21}b_{23}}{(b_{22}+b_{23}y)(b_{22}+b_{23}y^*)}+a_{22}\gamma_1](y-y^*)^2+(2\gamma_3a_2-\gamma_1\alpha_2)(u_2-u_2^*)(y-y^*).$$

在上式中取 $\gamma_1=\frac{a_2}{\alpha_2}$, $\gamma_2=\frac{\alpha_1}{2a_1}$, $\gamma_3=\frac{1}{2}$,则有:

$$D^+V_3(t)\leqslant[\frac{b_{11}}{b_{12}}+a_{12}(y^*+\varepsilon)-b_{14}]x-a_{11}x^2-2\gamma_2\eta_1u_1^2-2\gamma_3\eta_2(u_2-u_2^*)^2-$$
$$[\frac{\gamma_1b_{21}b_{23}}{(b_{22}+b_{23}y)(b_{22}+b_{23}y^*)}+a_{22}\gamma_1](y-y^*)^2\leqslant0,$$

当且仅当 $x=0$, $y=y^*$, $u_1=0$, $u_2=u_2^*$ 时取等号. 于是由 Lyapunov 稳定性定理[7] 可知,系统(2) 的边界平衡点$(0,y^*,0,u_2^*)$ 是全局稳定的. 证毕.

定理 4 设(x,y,u_1,u_2) 是系统(2) 的任意正解. 若 $b_{12}b_{14}>b_{11}$, $b_{22}b_{24}>b_{21}$,则系统(2) 的边界平衡点(0,0,0,0) 是全局稳定的.

证明 构造 Lyapunov 函数 $V_4(t)=c_1x+c_2y+c_3u_1^2+c_4u_2^2$,其中 $c_i(i=1,2,3,4)$ 是待定的正常数,于是有:

$$D^+V_4(t)=c_1x(\frac{b_{11}}{b_{12}+b_{13}x}-b_{14}-a_{11}x+a_{12}y-\alpha_1u_1)+$$
$$c_2y(\frac{b_{21}}{b_{22}+b_{23}y}-b_{24}-a_{22}y-\alpha_2u_2)+2c_3u_1(-\eta_1u_1+a_1x)+2c_4u_2(-\eta_2u_2+a_2y)\leqslant$$
$$c_1(\frac{b_{11}}{b_{12}}-b_{14})x-c_1a_{11}x^2+c_1a_{12}xy-c_1\alpha_1xu_1+c_2(\frac{b_{21}}{b_{22}}-b_{24})y-c_2a_{22}y^2-$$
$$c_2\alpha_2yu_2-2c_3\eta_1u_1^2+2c_3a_1xu_1-2c_4\eta_2u_2^2+2c_4a_2yu_2\leqslant$$
$$c_1(\frac{b_{11}}{b_{12}}-b_{14})x+c_2(\frac{b_{21}}{b_{22}}-b_{24})y-[c_1a_{11}x^2-c_1a_{12}xy+c_2a_{22}y^2]-$$
$$2c_3\eta_1u_1^2-2c_4\eta_2u_2^2+(2c_3a_1-c_1\alpha_1)xu_1+(2c_4a_2-c_2\alpha_2)yu_2.$$

在上式中取 $c_1=\frac{a_{11}a_{22}}{a_{12}^2}$, $c_2=\frac{1}{4}$, $c_3=\frac{c_1\alpha_1}{2a_1}$, $c_4=\frac{c_2\alpha_2}{2a_2}$,则有:

$$D^+V_4(t)\leqslant c_1(\frac{b_{11}}{b_{12}}-b_{14})x+c_2(\frac{b_{21}}{b_{22}}-b_{24})y-(\sqrt{c_1a_{11}}x-\sqrt{c_2a_{22}}y)^2-2c_3\eta_1u_1^2-2c_4\eta_2u_2^2.$$

上式表明 $\frac{\mathrm{d}V_4}{\mathrm{d}t}\leqslant0$,当且仅当 $x=y=u_1=u_2=0$ 时取等号. 于是由 Lyapunov 稳定性定理[7] 可知,系统(2) 的边界平衡点(0,0,0,0) 是全局稳定的. 证毕.

注 1 如果定理 1— 定理 4 的条件成立,则文献[6] 中定理 2.1 的条件也成立,由此可知加入反馈控制变量(系统(2)) 并未改变原系统(1) 的稳定性.

3 数值模拟

例 1 考虑如下系统:

$$\begin{cases}\dfrac{\mathrm{d}x}{\mathrm{d}t}=x(\dfrac{2}{1+x}-1-x+y-0.3u_1),\\ \dfrac{\mathrm{d}y}{\mathrm{d}t}=y(\dfrac{2}{1+y}-1-y-0.5u_2),\\ \dfrac{\mathrm{d}u_1}{\mathrm{d}t}=-0.8u_1+x,\\ \dfrac{\mathrm{d}u_2}{\mathrm{d}t}=-0.5u_2+0.8y,\end{cases}\tag{4}$$

其中 $b_{21}=2>b_{22}b_{24}=1$，$b_{11}=2>b_{12}b_{14}=1$，即系统(4)满足定理 1 的条件.由引理 1 和定理 1 知，系统(4)存在唯一的全局稳定的正平衡点(0.476，0.2995，0.595，0.4792).图 1 是系统(3)具有初值(1，0.3，0.1，0.2)，(0.4，2，0.3，0.7)，(0.02，2，1，0.5)，(1，2，0.2，1.5)的解的数值模拟图.

例 2　考虑如下系统：

$$\begin{cases}\dfrac{\mathrm{d}x}{\mathrm{d}t}=x(\dfrac{2}{1+x}-1-x+y-0.2u_1),\\ \dfrac{\mathrm{d}y}{\mathrm{d}t}=y(\dfrac{1}{2+y}-1-y-0.4u_2),\\ \dfrac{\mathrm{d}u_1}{\mathrm{d}t}=-0.6u_1+x,\\ \dfrac{\mathrm{d}u_2}{\mathrm{d}t}=-0.5u_2+0.7y,\end{cases}\tag{5}$$

其中 $b_{11}=2>b_{12}b_{14}=1$，$b_{22}b_{24}=2>b_{21}=1$，即系统(5)满足定理 2 的条件.由引理 2 和定理 2 知，系统(4)的边界平衡点(0.3561，0，0.5935，0)是全局稳定的.图 2 是系统(5)具有初值(1，0.3，0.1，0.2)，(0.4，1.1，0.3，0.7)，(0.02，0.8，1，0.5)，(1，0.6，0.2，0.2)的解的数值模拟图.

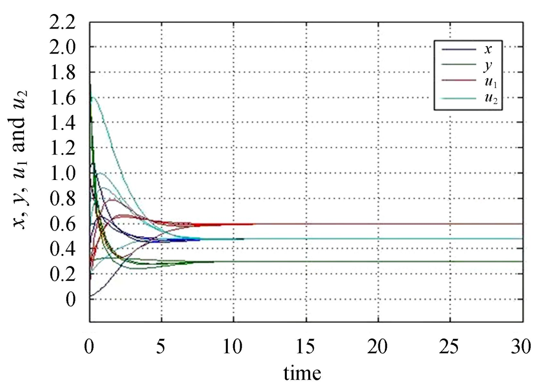

图 1　系统(4)的数值模拟图

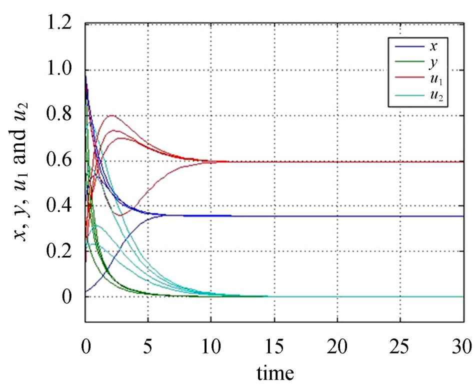

图 2　系统(5)的数值模拟图

例 3　考虑如下系统：

$$\begin{cases}\dfrac{\mathrm{d}x}{\mathrm{d}t}=x(\dfrac{1}{2+x}-1-x+y-0.4u_1),\\ \dfrac{\mathrm{d}y}{\mathrm{d}t}=y(\dfrac{2}{1+y}-1-y-0.6u_2),\\ \dfrac{\mathrm{d}u_1}{\mathrm{d}t}=-0.5u_1+x,\\ \dfrac{\mathrm{d}u_2}{\mathrm{d}t}=-0.6u_2+0.8y,\end{cases}\tag{6}$$

其中 $b_{21}=2>b_{22}b_{14}=1$，$\dfrac{b_{11}}{b_{12}}+a_{12}y^{*}=0.7995<b_{14}=1$，即系统(6)满足定理3的条件.由引理3和定理3知,系统(6)的边界平衡点(0,0.2950,0,0.3933)是全局稳定的.图3是系统(6)具有初值(1,0.3,0.1,0.2),(0.4,1.1,0.3,0.7),(0.02,0.8,1,0.5),(1,0.6,0.2,0.2)的解的数值模拟图.

例4 考虑如下系统:

$$\begin{cases}\dfrac{\mathrm{d}x}{\mathrm{d}t}=x\left(\dfrac{1}{2+x}-1-x+y-0.5u_1\right),\\ \dfrac{\mathrm{d}y}{\mathrm{d}t}=y\left(\dfrac{1}{2+y}-1-y-0.7u_2\right),\\ \dfrac{\mathrm{d}u_1}{\mathrm{d}t}=-0.6u_1+x,\\ \dfrac{\mathrm{d}u_2}{\mathrm{d}t}=-0.7u_2+0.8y,\end{cases}\tag{7}$$

其中 $b_{12}b_{14}=2>b_{11}=1$，$b_{22}b_{24}=2>b_{21}=1$，即系统(7)满足定理4的条件.由定理4知,系统(7)的边界平衡点(0,0,0,0)是全局稳定的.图4是系统(7)具有初值(1,0.3,0.1,0.2),(0.4,1.1,0.3,0.7),(0.02,0.8,1,0.5),(1,0.6,0.2,0.2)的解的数值模拟图.

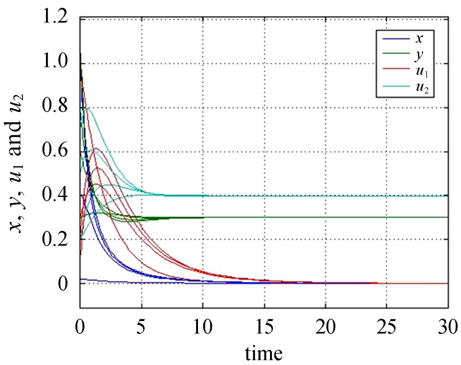

图3 系统(6)的数值模拟图

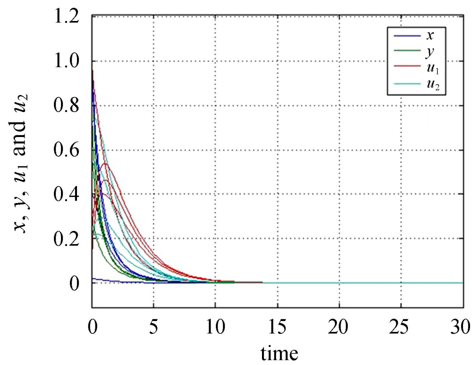

图4 系统(7)的数值模拟图

注2 由文献[6]中例3.1—例3.4可知,在系统(4)—系统(7)中未加入反馈控制变量时系统的相应平衡点是全局稳定的.由图1—图4可知,在系统(4)—系统(7)中加入反馈控制变量不会改变原系统平衡点的稳定性.

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