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Chern-Simons Landau-Lifshitz 模型自对偶方程静态解的存在性

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摘要: 研究了 Chern-Simons Landau-Lifshitz 模型自对偶方程静态解的存在性问题. 首先,利用数学分析理论和分离变量法得到了自对偶方程的静态解. 其次,证明了当向量场满足 $A_0=\phi_3-\tau$ 时自对偶方程的静态解满足 Chern-Simons Landau-Lifshitz 方程. 最后,利用共变导数和向量运算法则证明了 Chern-Simons Landau-Lifshitz 模型具有能量守恒和规范不变的性质.

关键词: Chern-Simons Landau-Lifshitz 模型; 分离变量法; 静态解
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Study on the existence of static solutions to the self-dual equation for the Chern-Simons Landau-Lifshitz model

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Abstract: This paper considered the existence of static solutions to the self-dual case of the Chern-Simons Landau-Lifshitz model. First, the static solutions of the self-dual equations are obtained using the mathematical analysis technique and the separation variable method. Secondly, it is shown that the static solution of the self-dual equation also satisfies the Chern-Simons Landau-Lifshitz equation when the vector field is satisfied $A_0=\phi_3-\tau$. Finally, we use the covariant derivative and the vector operation rules to show that the Chern-Simons Landau-Lifshitz model has both energy-conservation and gauge-invariant properties.

Keywords: Chern-Simons Landau-Lifshitz model; separation of variables; static solutions

0 引言

本文考虑如下 Chern-Simons Landau-Lifshitz 模型的自对偶方程静态解的存在性问题:

$$D_0\boldsymbol{\phi}=\boldsymbol{\phi}\times D_1D_1\boldsymbol{\phi}-N^2\phi_3(\boldsymbol{n}\times\boldsymbol{\phi})-V'(\tau-\boldsymbol{n}\cdot\boldsymbol{\phi})(\boldsymbol{n}\times\boldsymbol{\phi}),\tag{1}$$

$$F_{01}=N|\boldsymbol{n}\times\boldsymbol{\phi}|^2,\tag{2}$$

$$\partial_0 N=-\langle\boldsymbol{n}\times\boldsymbol{\phi},D_1\boldsymbol{\phi}\rangle,\tag{3}$$

$$\partial_1 N=\tau-\phi_3.\tag{4}$$

其中: $\boldsymbol{\phi}=\{(\phi_1,\phi_2,\phi_3):\mathbf{R}\to S^2=(s_1,s_2,s_3)\,|\,s_1^2+s_2^2+s_3^2=1\}$ 是单位球体中的自旋向量,且 $|\boldsymbol{\phi}|=1$;

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$A_\mu:\mathbf{R}^{2+1}\to\mathbf{R}$ 是向量场;$\mu=0,1$;$\boldsymbol{n}=(0,0,1)$;$D_\mu\boldsymbol{\phi}$ 是共变导数,$D_\mu\boldsymbol{\phi}=\partial_\mu\boldsymbol{\phi}+A_\mu(\boldsymbol{n}\times\boldsymbol{\phi})$;$F_{\mu\nu}$ 是曲率项,$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$;$V(\tau-\boldsymbol{n}\cdot\boldsymbol{\phi})$ 是位势项,$V(\tau-\boldsymbol{n}\cdot\boldsymbol{\phi})=\kappa(\tau-\boldsymbol{n}\cdot\boldsymbol{\phi})^2$,$\kappa$ 是 Chern-Simons 耦合常数($\kappa>0$),表示 Chern-Simons 作用强度. 当 $\kappa=1$ 时,该系统的自对偶方程为:

$$D_1\boldsymbol{\phi}+N\boldsymbol{\phi}\times(\boldsymbol{n}\times\boldsymbol{\phi})=0,\tag{5}$$

$$\partial_1 N=\tau-\phi_3.\tag{6}$$

Landau-Lifshitz 方程是一类重要的非线性偏微分方程,由于其可描述磁性物质动态磁化现象,因此在铁磁物质的动态磁化理论研究和应用中具有重要作用[1-5]. Chern-Simons Landau-Lifshitz 方程(CSLL 方程)在凝聚态物理、材料功能的应用等方面具有重要应用价值,但由于 CSLL 方程存在 Chern-Simons 规范场的耦合,因此对其研究存在较多困难. 近年来,一些学者在适当假设的前提下对该方程的行波解、爆破解及解的渐进行为等问题进行了研究[6-10]. 本文基于上述研究,研究了 CSLL 方程自对偶情况下的静态解,并通过计算证明了 CSLL 方程具有能量守恒和规范不变的性质.

1　主要结果及其证明

定理 1　假设 $A_1=0$,则自对偶方程(5) 和方程(6) 有显式静态解:

$$\phi_1(x)=C_1\exp\left\{\int_0^x\frac{1-F^{-1}(z)}{1+F^{-1}(z)}\left((\tau+1)z-\int_0^z\frac{2}{1+F^{-1}(y)}\mathrm{d}y\right)\mathrm{d}z\right\},$$

$$\phi_2(x)=C_2\exp\left\{\int_0^x\frac{1-F^{-1}(z)}{1+F^{-1}(z)}\left((\tau+1)z-\int_0^z\frac{2}{1+F^{-1}(y)}\mathrm{d}y\right)\mathrm{d}z\right\},$$

$$\phi_3(x)=\frac{2}{1+F^{-1}(x)}-1,\quad N(x)=(\tau+1)x-\int_0^x\frac{2}{1+F^{-1}(y)}\mathrm{d}y+C.$$

其中 C、C_1 和 C_2 都是常数,且满足关系 $C_1=kC_2$.

证明　首先假设自对偶方程(5) 的静态解的形式为:

$$\boldsymbol{\phi}(t,x)=(\phi_1(x),\phi_2(x),\phi_3(x)),\ N(t,x)=N(x).\tag{7}$$

则当 $A_1=0$ 时,由式(3) 有 $0=\langle\boldsymbol{n}\times\boldsymbol{\phi},D_1\boldsymbol{\phi}\rangle=\langle\boldsymbol{n}\times\boldsymbol{\phi},\partial_1\boldsymbol{\phi}+A_1(\boldsymbol{n}\times\boldsymbol{\phi})\rangle=\langle\boldsymbol{n}\times\boldsymbol{\phi},\partial_1\boldsymbol{\phi}\rangle$. 将方程(5) 两端与向量 $\boldsymbol{n}\times\boldsymbol{\phi}$ 做内积可得:

$$0=\langle\boldsymbol{n}\times\boldsymbol{\phi},\partial_1\boldsymbol{\phi}\rangle+A_1\langle\boldsymbol{n}\times\boldsymbol{\phi},\boldsymbol{n}\times\boldsymbol{\phi}\rangle+N\langle\boldsymbol{n}\times\boldsymbol{\phi},\boldsymbol{\phi}\times(\boldsymbol{n}\times\boldsymbol{\phi})\rangle=$$
$$\langle\boldsymbol{n}\times\boldsymbol{\phi},\partial_1\boldsymbol{\phi}\rangle+A_1|\boldsymbol{n}\times\boldsymbol{\phi}|^2+N\langle\boldsymbol{\phi},(\boldsymbol{n}\times\boldsymbol{\phi})\times(\boldsymbol{n}\times\boldsymbol{\phi})\rangle=\langle\boldsymbol{n}\times\boldsymbol{\phi},\partial_1\boldsymbol{\phi}\rangle+A_1|\boldsymbol{n}\times\boldsymbol{\phi}|^2.$$

由上式可知假设 $A_1=0$ 是合理的. 将 $A_1=0$ 代入方程(5) 可得:

$$D_1\boldsymbol{\phi}+N\boldsymbol{\phi}\times(\boldsymbol{n}\times\boldsymbol{\phi})=\partial_1\boldsymbol{\phi}+N(\boldsymbol{n}-\boldsymbol{\phi}\langle\boldsymbol{n},\boldsymbol{\phi}\rangle)=\partial_1\boldsymbol{\phi}+N(\boldsymbol{n}-\phi_3\boldsymbol{\phi}).$$

根据上式可将方程(5) 和方程(6) 化简为如下形式:

$$\phi_1'-N\phi_1\phi_3=0,\tag{8}$$

$$\phi_2'-N\phi_2\phi_3=0,\tag{9}$$

$$\phi_3'+N(1-\phi_3^2)=0,\tag{10}$$

$$N'=\tau-\phi_3.\tag{11}$$

显然,求解上述方程组的关键是求解 $\phi_3(x)$. 假设:

$$\phi_1(x)=\frac{2u_1(x)}{1+|u(x)|^2},\ \phi_2=\frac{2u_2(x)}{1+|u(x)|^2},\ \phi_3=\frac{1-|u(x)|^2}{1+|u(x)|^2},\tag{12}$$

其中 $|u(x)|^2=u_1^2(x)+u_2^2(x)$. 据此对 $\phi_3(x)$ 进行求导可得:

$$\phi_3'=\frac{-2uu'(1+|u|^2)-(1-|u|^2)2uu'}{(1+|u|^2)^2}=\frac{-4uu'}{(1+|u|^2)^2}.$$

由此可将方程(10)化简为 $N=\dfrac{uu'}{|u|^2}=\dfrac{1}{2}(\ln|u|^2)'$，于是再由方程(11)可得：

$$\frac{1}{2}(\ln|u|^2)''=N'=\tau-\phi_3=\tau-\frac{1-|u|^2}{1+|u|^2}=\frac{(\tau-1)+(\tau+1)|u|^2}{1+|u|^2}. \tag{13}$$

定义函数 $h(x):=\dfrac{\tau-1+(\tau+1)x}{1+x}$，其中 $x\geqslant 0$. 由此易知当 $0\leqslant x\leqslant\dfrac{1-\tau}{1+\tau}$ 时，$h(x)\leqslant 0$；当 $x\geqslant\dfrac{1-\tau}{1+\tau}$ 时，$h(x)\geqslant 0$. 当 $\tau=1$ 时，$h(x)\geqslant 0$，$x\geqslant 0$，且 $\phi=(0,0,1)$，于是根据式(12)可得 $u=0$. 下面假设 $\tau<1$，并记 $\rho:=|u|^2$，由此得 $(\ln\rho)'=\dfrac{\rho'}{\rho}$. 据此方程(13)可进一步表示为

$$\frac{1}{2}(\ln\rho)''=\frac{1}{2}\left(\frac{\rho'}{\rho}\right)'=\frac{(\tau-1)+(\tau+1)\rho}{1+\rho}.$$

将该等式两端同时乘 $\dfrac{\rho'}{\rho}$ 后可得：

$$\left(\frac{1}{4}\left(\frac{\rho'}{\rho}\right)^2\right)'=\frac{1}{2}\left(\frac{\rho'}{\rho}\right)'\frac{\rho'}{\rho}=\frac{(\tau-1)+(\tau+1)\rho}{(1+\rho)\rho}\rho'=\left(\frac{2}{1+\rho}+\frac{\tau-1}{\rho}\right)\rho'=$$
$$(\ln(1+\rho)^2+(\tau-1)\ln\rho)'.$$

将上式左右两端同时积分并整理后可得：

$$\frac{1}{4}\left(\frac{\rho'}{\rho}\right)^2=\ln\frac{(1+\rho)^2}{\rho^{1-\tau}}+c, \tag{14}$$

其中 c 是任意常数. 值得注意的是，当 $\phi_3\to\tau$ 时，根据方程(11)和方程(13)可知 $N'\to 0$ 且 $h(x)\to 0$，即 $\rho=|u|^2\to\dfrac{1-\tau}{1+\tau}$. 假设 $0\leqslant\tau<1$，函数 $f(\rho):=\dfrac{(1+\rho)^2}{\rho^{1-\tau}}$，则对 $f(\rho)$ 求导可得：

$$f'(\rho)=\frac{2(1+\rho)\rho^{1-\tau}-(1-\tau)(1+\rho)^2\rho^{-\tau}}{\rho^{2-2\tau}}=\frac{1+\rho}{\rho^{2-\tau}}((\tau-1)+(\tau+1)\rho).$$

显然，当 $0<\rho<\dfrac{1-\tau}{1+\tau}$ 时，$f'(\rho)<0$；当 $\rho>\dfrac{1-\tau}{1+\tau}$ 时，$f'(\rho)>0$；当 $\rho=\dfrac{1-\tau}{1+\tau}$ 时，$f'(\rho)=0$. 由此可知函数 $f(\rho)$ 在 $\rho=\dfrac{1-\tau}{1+\tau}$ 处应取最小值，且 $\min\limits_{\rho=\frac{1-\tau}{1+\tau}}f(\rho)=\dfrac{4}{(1-\tau)^{1-\tau}(1+\tau)^{1+\tau}}$. 令常数 $c=-\ln f\left(\dfrac{1-\tau}{1+\tau}\right)=\ln\dfrac{(1-\tau)^{1-\tau}(1+\tau)^{1+\tau}}{4}$，由此知方程(14)右端恒大于零. 现考虑如下初值问题：

$$\begin{cases}\rho'(x)=2\rho\left(\ln H(\tau)\dfrac{(1+\rho)^2}{\rho^{1-\tau}}\right)^{\frac{1}{2}},\ x\geqslant 0;\\ \rho(0)=0.\end{cases} \tag{15}$$

其中 $H(\tau):=\dfrac{(1-\tau)^{1-\tau}(1+\tau)^{1+\tau}}{4}$. 定义函数

$$g(\rho)=\rho\left(\ln H(\tau)\frac{(1+\rho)^2}{\rho^{1-\tau}}\right)^{\frac{1}{2}}=\rho\left(\ln\frac{(1+\rho)^2}{\rho^{1-\tau}}-\ln\frac{4}{(1-\tau)^{1-\tau}(1+\tau)^{1+\tau}}\right)^{\frac{1}{2}},$$

则当 $0<\rho<\dfrac{1-\tau}{1+\tau}$ 时，$g(\rho)>0$ 且 $g(0)=g\left(\dfrac{1-\tau}{1+\tau}\right)=0$. 对 $g(\rho)$ 函数求导可得：

$$g'(\rho)=(\ln H(\tau)f(\rho))^{-\frac{1}{2}}+\frac{\rho}{2}(\ln H(\tau)f(\rho))^{-\frac{1}{2}}\frac{f'(\rho)}{f(\rho)}=$$
$$(\ln H(\tau)f(\rho))^{-\frac{1}{2}}\left(\ln H(\tau)f(\rho)+\frac{\rho}{2}\frac{(\tau+1)\rho+(\tau-1)}{(1+\rho)\rho}\right)=$$

$$\left(\ln H(\tau)\frac{(1+\rho)^2}{\rho^{1-\tau}}\right)^{-\frac{1}{2}}\left(\ln H(\tau)\frac{(1+\rho)^2}{\rho^{1-\tau}}+\frac{(\tau+1)\rho-(1-\tau)}{2(1+\rho)}\right).$$

由于 $\tau<1$，因此 $\lim\limits_{\rho\to 0^+} f(\rho)=\infty$，$\lim\limits_{\rho\to 0^+}(\ln H(\tau)f(\rho))^{-\frac{1}{2}}=0$，$\lim\limits_{\rho\to 0^+}\frac{(1-\tau)\rho-(1-\tau)}{2(1+\rho)}<\infty$，故 $\lim\limits_{\rho\to 0^+} g'(\rho)=\infty$. 再根据 $H(\tau)$ 的定义可知，当 $\rho=\frac{1-\tau}{1+\tau}$ 时，$\ln H(\tau)\frac{(1+\rho)^2}{\rho^{1-\tau}}=0$，于是有

$$\lim_{\rho\to\left(\frac{1-\tau}{1+\tau}\right)^-} g'(\rho)=\lim_{\rho\to\left(\frac{1-\tau}{1+\tau}\right)^-}\left(\ln H(\tau)\frac{(1+\rho)^2}{\rho^{1-\tau}}\right)^{-\frac{1}{2}}\frac{(\tau+1)\rho-(1-\tau)}{2(1+\rho)}.$$

若考虑 $0<\rho<1$，则有 $\rho^{1+\tau}<\rho^{1-\tau}<\rho^{-\tau}$，因此 $\ln H(\tau)\frac{(1+\rho)^2}{\rho^{-\tau}}<\ln H(\tau)\frac{(1+\rho)^2}{\rho^{1-\tau}}<\ln H(\tau)\cdot\frac{(1+\rho)^2}{\rho^{1+\tau}}$. 又因为 $\lim\limits_{\rho\to\left(\frac{1-\tau}{1+\tau}\right)^-}\left(\ln H(\tau)\frac{(1+\rho)^2}{\rho^{-\tau}}\right)^{-\frac{1}{2}}=\left(\ln\frac{(1-\tau)^{1-\tau}(1+\tau)^{1+\tau}}{4}+\ln\frac{\left(1+\frac{1-\tau}{1+\tau}\right)^2}{\left(\frac{1-\tau}{1+\tau}\right)^{-\tau}}\right)^{-\frac{1}{2}}=\left(\ln\frac{1-\tau}{1+\tau}\right)^{-\frac{1}{2}}$，故可求得：

$$\lim_{\rho\to\left(\frac{1-\tau}{1+\tau}\right)^-}\left(\ln H(\tau)\frac{(1+\rho)^2}{\rho^{-\tau}}\right)^{-\frac{1}{2}}\frac{(\tau+1)\rho-(1-\tau)}{2(1+\rho)}=0.$$

同理可得：

$$\lim_{\rho\to\left(\frac{1-\tau}{1+\tau}\right)^-}\left(\ln H(\tau)\frac{(1+\rho)^2}{\rho^{1+\tau}}\right)^{-\frac{1}{2}}=\left(\ln\frac{(1-\tau)^{1-\tau}(1+\tau)^{1+\tau}}{4}+\ln\frac{\left(1+\frac{1-\tau}{1+\tau}\right)^2}{\left(\frac{1-\tau}{1+\tau}\right)^{1+\tau}}\right)^{-\frac{1}{2}}=\left(\ln\left(\frac{1+\tau}{1-\tau}\right)^{2\tau}\right)^{-\frac{1}{2}},$$

故有 $\lim\limits_{\rho\to\left(\frac{1-\tau}{1+\tau}\right)^-}\left(\ln H(\tau)\frac{(1+\rho)^2}{\rho^{1+\tau}}\right)^{-\frac{1}{2}}\frac{(\tau+1)\rho-(1-\tau)}{2(1+\rho)}=0$. 再由夹逼定理可知：

$$\lim_{\rho\to\left(\frac{1-\tau}{1+\tau}\right)^-} g'(\rho)=\lim_{\rho\to\left(\frac{1-\tau}{1+\tau}\right)^-}\left(\ln H(\tau)\frac{(1+\rho)^2}{\rho^{1-\tau}}\right)^{-\frac{1}{2}}\frac{(\tau+1)\rho-(1-\tau)}{2(1+\rho)}=0.$$

由于导数表示变化率，因此结合上述两个极限可知：

$$\int_0^{\varepsilon}\frac{1}{g'(\rho)}\,d\rho=M<\infty,\ \int_{\frac{1-\tau}{1+\tau}-\varepsilon}^{\frac{1-\tau}{1+\tau}}\frac{1}{g'(\rho)}\,d\rho=\infty. \tag{16}$$

对初值问题(15)进行积分，并定义函数 $F(\rho):=\int_0^{\rho}\frac{1}{\omega}\,d\omega$，则根据反函数推导定理可知，$F(\rho)=x$. 再由式(16)知 $F(0)=0$，$\lim\limits_{\rho\to\left(\frac{1-\tau}{1+\tau}\right)^-}F(\rho)=\infty$. 由于导函数 $F'(\rho)=\frac{1}{2g(\rho)}>0$，并且 $g(0)=g\left(\frac{1-\tau}{1+\tau}\right)=0$，所以有 $\lim\limits_{\rho\to 0^+}F'(\rho)=\lim\limits_{\rho\to\left(\frac{1-\tau}{1+\tau}\right)^-}F'(\rho)=\infty$. 定义函数 $G(x):=F^{-1}(x)=\rho$，其中 $x\geqslant 0$，且 $G(x)$ 满足 $G(0)=0$，$\lim\limits_{x\to\infty}G(x)=\frac{1-\tau}{1+\tau}$，$G'(0)=\frac{1}{F'(\rho)}=2g(\rho)=0$. 将 $\rho=|u|^2$ 代入作为变量替换的式(12)中可得 $\phi_3=\frac{1-|u|^2}{1+|u|^2}=\frac{2}{1+F^{-1}(x)}-1$，再将 ϕ_3 代入到方程(11)中可得：

$$N(x)=\int_0^x\left((\tau+1)-\frac{2}{1+F^{-1}(y)}\right)dy+C=(\tau+1)x-\int_0^x\frac{2}{1+F^{-1}(y)}\,dy+C.$$

由方程(8)可得 $\frac{\phi_1'}{\phi_1}=N\phi_3$. 对该等式利用分离变量法可解得：

$$\phi_1=C_1\exp\left(\int_0^x N\phi_3\,\mathrm{d}z\right)=C_1\exp\left(\int_0^x \frac{1-F^{-1}(z)}{1+F^{-1}(z)}\left((\tau+1)z-\int_0^z \frac{2}{1+F^{-1}(y)}\,\mathrm{d}y\right)\mathrm{d}z\right).$$

同理,由方程(9)可得等式 $\dfrac{\phi_2'}{\phi_2}=N\phi_3$. 对该等式利用分离变量法可解得:

$$\phi_2=C_2\exp\left(\int_0^x N\phi_3\,\mathrm{d}y\right)=C_2\exp\left(\int_0^x \frac{1-F^{-1}(z)}{1+F^{-1}(z)}\left((\tau+1)z-\int_0^z \frac{2}{1+F^{-1}(y)}\,\mathrm{d}y\right)\mathrm{d}z\right).$$

再由方程(8)和方程(9)可知 ϕ_1 和 ϕ_2 满足 $\phi_1=k\phi_2$, 因此 $C_1=kC_2$. 定理 1 得证.

定理 2 若 $A_0=\phi_3-\tau$, 则方程(5)和方程(6)的静态解 $\boldsymbol{\phi}=(\phi_1(x),\phi_2(x),\phi_3(x))$, $N=N(x)$ 是系统(1)—(4)的解.

证明 根据方程(5)可知 $D_1\boldsymbol{\phi}=-N\boldsymbol{\phi}\times(\boldsymbol{n}\times\boldsymbol{\phi})=-N\boldsymbol{n}+N\phi_3\boldsymbol{\phi}$, 由此得 $\boldsymbol{n}\times D_1\boldsymbol{\phi}=\boldsymbol{n}\times(-N\boldsymbol{n}+N\phi_3\boldsymbol{\phi})=N\phi_3(\boldsymbol{n}\times\boldsymbol{\phi})$. 又因为 $D_\alpha(\boldsymbol{\phi}\times\boldsymbol{\psi})=(D_\alpha\boldsymbol{\phi}\times\boldsymbol{\psi})+(\boldsymbol{\phi}\times D_\alpha\boldsymbol{\psi})$, 所以有:

$$\boldsymbol{\phi}\times D_1D_1\boldsymbol{\phi}=D_1(\boldsymbol{\phi}\times D_1\boldsymbol{\phi})=D_1(\boldsymbol{\phi}\times(-N\boldsymbol{n}+N\phi_3\boldsymbol{\phi}))=D_1(N(\boldsymbol{n}\times\boldsymbol{\phi})).$$

由于 $A_0=\phi_3-\tau$, $A_1=0$, 则 $\boldsymbol{\phi}\times D_1D_1\boldsymbol{\phi}=\partial_1N(\boldsymbol{n}\times\boldsymbol{\phi})-N\partial_1(\boldsymbol{n}\times\boldsymbol{\phi})=(\tau-\phi_3)(\boldsymbol{n}\times\boldsymbol{\phi})-N(\partial_1\boldsymbol{\phi}\times\boldsymbol{n})=-A_0(\boldsymbol{n}\times\boldsymbol{\phi})+N^2\phi_3(\boldsymbol{n}\times\boldsymbol{\phi})$. 又因为 $V(\tau-\boldsymbol{n}\cdot\boldsymbol{\phi})=(\tau-\phi_3)^2$, 则 $V'(\tau-\boldsymbol{n}\cdot\boldsymbol{\phi})=2(\tau-\phi_3)=-2A_0$, 故有 $\boldsymbol{\phi}\times D_1D_1\boldsymbol{\phi}-N^2\phi_3(\boldsymbol{n}\times\boldsymbol{\phi})-V'(\tau-\boldsymbol{n}\cdot\boldsymbol{\phi})(\boldsymbol{n}\times\boldsymbol{\phi})=-A_0(\boldsymbol{n}\times\boldsymbol{\phi})+2A_0(\boldsymbol{n}\times\boldsymbol{\phi})=A_0(\boldsymbol{n}\times\boldsymbol{\phi})=D_0\boldsymbol{\phi}$. 因此方程(1)成立. 再由方程(5)可知, $F_{01}=-\partial_1A_0=-\partial_1\phi_3=-\langle\boldsymbol{n},D_1\boldsymbol{\phi}\rangle=-\langle\boldsymbol{n},-N\boldsymbol{n}+N\phi_3\boldsymbol{\phi}\rangle=N-N\phi_3\langle\boldsymbol{n},\boldsymbol{\phi}\rangle=N(1-\phi_3^2)=N|\boldsymbol{n}\times\boldsymbol{\phi}|^2$, 因此方程(2)成立. 将方程(5)代入可得 $-\langle\boldsymbol{n}\times\boldsymbol{\phi},D_1\boldsymbol{\phi}\rangle=-\langle\boldsymbol{n}\times\boldsymbol{\phi},-N\boldsymbol{n}+N\phi_3\boldsymbol{\phi}\rangle=N\langle\boldsymbol{n}\times\boldsymbol{\phi},\boldsymbol{n}\rangle-N\phi_3\langle\boldsymbol{n}\times\boldsymbol{\phi},\boldsymbol{\phi}\rangle=0=\partial_0N$, 因此方程(3)成立. 显然式(4)也成立. 综上, 定理 2 得证.

2 性质

性质 1 系统(1)—(4)有如下守恒能量:

$$E(t)=\int_{\mathbf{R}}\frac{1}{2}|D_1\boldsymbol{\phi}|^2+\frac{1}{2}N^2|\boldsymbol{n}\times\boldsymbol{\phi}|^2+V(\tau-\boldsymbol{n}\cdot\boldsymbol{\phi})\mathrm{d}x=E(0). \tag{17}$$

证明 首先将方程(1)与 $\boldsymbol{\phi}$ 做外积, 然后再与 $D_0\boldsymbol{\phi}$ 做内积, 则方程(1)可化简为

$$\begin{aligned}0={}&\langle\boldsymbol{\phi}\times(\boldsymbol{\phi}\times D_1D_1\boldsymbol{\phi}),D_0\boldsymbol{\phi}\rangle-N^2\phi_3\langle\boldsymbol{\phi}\times(\boldsymbol{n}\times\boldsymbol{\phi}),D_0\boldsymbol{\phi}\rangle-\\&V'(\tau-\boldsymbol{n}\cdot\boldsymbol{\phi})\langle\boldsymbol{\phi}\times(\boldsymbol{n}\times\boldsymbol{\phi}),D_0\boldsymbol{\phi}\rangle=\langle\boldsymbol{\phi},D_0\boldsymbol{\phi}\rangle\langle\boldsymbol{\phi},D_1D_1\boldsymbol{\phi}\rangle-\langle D_1D_1\boldsymbol{\phi},D_0\boldsymbol{\phi}\rangle-\\&N^2\phi_3\langle\boldsymbol{n},D_0\boldsymbol{\phi}\rangle+N^2\phi_3^2\langle\boldsymbol{\phi},D_0\boldsymbol{\phi}\rangle-V'(\tau-\phi_3)\langle\boldsymbol{n},D_0\boldsymbol{\phi}\rangle+V'(\tau-\phi_3)\phi_3\langle\boldsymbol{\phi},D_0\boldsymbol{\phi}\rangle.\end{aligned}$$

根据共轭导数的定义可得 $\langle\boldsymbol{n},D_0\boldsymbol{\phi}\rangle=\langle\boldsymbol{n},\partial_0\boldsymbol{\phi}\rangle+A_0\langle\boldsymbol{n},\boldsymbol{n}\times\boldsymbol{\phi}\rangle=\partial_0\phi_3$, $\langle\boldsymbol{\phi},D_0\boldsymbol{\phi}\rangle=0$, 因此上式可化简为 $0=-\langle D_1D_1\boldsymbol{\phi},D_0\boldsymbol{\phi}\rangle-N^2\phi_3\partial_0\phi_3-V'(\tau-\phi_3)\partial_0\phi_3$. 利用公式 $D_\alpha D_\beta\boldsymbol{\phi}=D_\beta D_\alpha\boldsymbol{\phi}+F_{\alpha\beta}(\boldsymbol{n}\times\boldsymbol{\phi})$ 及 $\partial_\alpha\langle\boldsymbol{\phi},\boldsymbol{\psi}\rangle=\langle D_\alpha\boldsymbol{\phi},\boldsymbol{\psi}\rangle+\langle\boldsymbol{\phi},D_\alpha\boldsymbol{\psi}\rangle$ 对上式进行计算可得:

$$\begin{aligned}0={}&-\partial_1\langle D_1\boldsymbol{\phi},D_0\boldsymbol{\phi}\rangle+\langle D_1\boldsymbol{\phi},D_1D_0\boldsymbol{\phi}\rangle+\frac{1}{2}N^2\partial_0(1-\phi_3^2)+\partial_0(V(\tau-\phi_3))=\\&-\partial_1\langle D_1\boldsymbol{\phi},D_0\boldsymbol{\phi}\rangle+\langle D_1\boldsymbol{\phi},D_0D_1\boldsymbol{\phi}\rangle+F_{10}\langle D_1\boldsymbol{\phi},\boldsymbol{n}\times\boldsymbol{\phi}\rangle+\frac{1}{2}N^2\partial_0(|\boldsymbol{n}\times\boldsymbol{\phi}|^2)+\\&\partial_0(V(\tau-\phi_3))=-\partial_1\langle D_1\boldsymbol{\phi},D_0\boldsymbol{\phi}\rangle+\frac{1}{2}\partial_0\langle D_1\boldsymbol{\phi},D_1\boldsymbol{\phi}\rangle+N|\boldsymbol{n}\times\boldsymbol{\phi}|^2\partial_0N+\\&\frac{1}{2}N^2\partial_0(|\boldsymbol{n}\times\boldsymbol{\phi}|^2)+\partial_0(V(\tau-\phi_3))=-\partial_1\langle D_1\boldsymbol{\phi},D_0\boldsymbol{\phi}\rangle+\frac{1}{2}\partial_0(|D_1\boldsymbol{\phi}|^2)+\\&\frac{1}{2}\partial_0(N^2|\boldsymbol{n}\times\boldsymbol{\phi}|^2)+\partial_0(V(\tau-\phi_3)).\end{aligned}$$

将上式两端同时在 **R** 上积分可得:

$$0=-\int_{\mathbf{R}}\partial_1\langle D_1\boldsymbol{\phi},D_0\boldsymbol{\phi}\rangle\mathrm{d}x+\int_{\mathbf{R}}\partial_0\left(\frac{1}{2}|D_1\boldsymbol{\phi}|^2+\frac{1}{2}N^2|\boldsymbol{n}\times\boldsymbol{\phi}|^2+V(\tau-\phi_3)\right)\mathrm{d}x=$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{R}}\frac{1}{2}|D_1\boldsymbol{\phi}|^2+\frac{1}{2}N^2|\boldsymbol{n}\times\boldsymbol{\phi}|^2+V(\tau-\phi_3)\mathrm{d}x,$$

因此系统(1)—(4) 有如式(17) 的守恒能量.

性质 2　系统(1)—(4) 在如下规范变化下保持不变:

$$\boldsymbol{\phi}=(z,\phi_3)\rightarrow\widetilde{\boldsymbol{\phi}}=(\tilde{z}=z\mathrm{e}^{\mathrm{i}\chi},\tilde{\phi}_3=\phi_3),\ A_\mu\rightarrow\widetilde{A}_\mu=A_\mu-\partial_\mu\chi,\ N\rightarrow N,\tag{18}$$

其中 $z=\phi_1+\mathrm{i}\phi_2$, 函数 $\chi\in C^\infty(\mathbf{R}^{1+1})$.

证明　首先考虑方程(1) 的规范不变性. 记 $z=\phi_1+\mathrm{i}\phi_2$ 经过规范变换后为 $\tilde{z}=\tilde{\phi}_1+\mathrm{i}\tilde{\phi}_2$, 则式(18) 可表示为:

$$\boldsymbol{\phi}=(\phi_1,\phi_2,\phi_3)\rightarrow\widetilde{\boldsymbol{\phi}}=(\tilde{\phi}_1=\phi_1\cos\chi-\phi_2\sin\chi,\tilde{\phi}_2=\phi_1\sin\chi+\phi_2\cos\chi,\tilde{\phi}_3=\phi_3),$$

$$A_\mu\rightarrow\widetilde{A}_\mu=A_\mu-\partial_\mu\chi,\ N\rightarrow N.$$

由外积定义可知 $\boldsymbol{n}\times\boldsymbol{\phi}=(\mathrm{i}z,0)$, $\boldsymbol{n}\times\widetilde{\boldsymbol{\phi}}=(\mathrm{i}\tilde{z},0)=(\mathrm{i}z\mathrm{e}^{\mathrm{i}\chi},0)$. 再对 $N^2\phi_3(\boldsymbol{n}\times\boldsymbol{\phi})=N^2\phi_3(\mathrm{i}z,0)$ 进行变换可得 $\widetilde{N}^2\tilde{\phi}_3(\boldsymbol{n}\times\widetilde{\boldsymbol{\phi}})=N^2\phi_3(\mathrm{i}z\mathrm{e}^{\mathrm{i}\chi},0)$. 同理, 对 $V'(\tau-\boldsymbol{n}\cdot\boldsymbol{\phi})(\boldsymbol{n}\times\boldsymbol{\phi})=V'(\tau-\phi_3)(\mathrm{i}z,0)$ 项进行变换后可得:

$$V'(\tau-\boldsymbol{n}\cdot\widetilde{\boldsymbol{\phi}})(\boldsymbol{n}\times\widetilde{\boldsymbol{\phi}})=V'(\tau-\phi_3)(\mathrm{i}z\mathrm{e}^{\mathrm{i}\chi},0).$$

再根据共变导数的定义可知:

$$D_0\boldsymbol{\phi}=\partial_0\boldsymbol{\phi}+A_0(\boldsymbol{n}\times\boldsymbol{\phi})=\partial_0(z,\phi_3)+A_0(\mathrm{i}z,0)=(\partial_0 z+\mathrm{i}A_0 z,\partial_0\phi_3).$$

类似上述方法可得:

$$D_0\widetilde{\boldsymbol{\phi}}=(\partial_0\tilde{z}+\mathrm{i}\widetilde{A}_0\tilde{z},\partial_0\phi_3)=(\partial_0(z\mathrm{e}^{\mathrm{i}\chi})+\mathrm{i}(A_0-\partial_0\chi)z\mathrm{e}^{\mathrm{i}\chi},\partial_0\phi_3)=((\partial_0 z+\mathrm{i}A_0 z)\mathrm{e}^{\mathrm{i}\chi},\partial_0\phi_3).$$

综上可知, 只需要证明 $\boldsymbol{\phi}\times D_1D_1\boldsymbol{\phi}=(\alpha+\mathrm{i}\beta,\gamma)$, 且对 $\boldsymbol{\phi}\times D_1D_1\boldsymbol{\phi}$ 进行变换后有 $\widetilde{\boldsymbol{\phi}}\times D_1D_1\widetilde{\boldsymbol{\phi}}=((\alpha+\mathrm{i}\beta)\mathrm{e}^{\mathrm{i}\chi},\gamma)$, 即可证明方程(1) 满足规范不变性.

类似 $D_0\boldsymbol{\phi}$ 和 $D_0\widetilde{\boldsymbol{\phi}}$ 的计算, 易得 $D_1\boldsymbol{\phi}=(\partial_1 z+\mathrm{i}A_1 z,\partial_1\phi_3)$, $D_1\widetilde{\boldsymbol{\phi}}=((\partial_1 z+\mathrm{i}A_1 z)\mathrm{e}^{\mathrm{i}\chi},\partial_1\phi_3)$. 由此经进一步计算可得:

$$\begin{aligned}&D_1D_1\boldsymbol{\phi}=\partial_1(D_1\boldsymbol{\phi})+A_1(\boldsymbol{n}\times D_1\boldsymbol{\phi})=\partial_1(\partial_1 z+\mathrm{i}A_1 z,\partial_1\phi_3)+A_1(-A_1 z+\mathrm{i}\partial_1 z,0)=\\&\quad(\partial_1\partial_1 z+\mathrm{i}z\partial_1 A_1+\mathrm{i}A_1\partial_1 z,\partial_1\partial_1\phi_3)+(-A_1^2 z+\mathrm{i}A_1\partial_1 z,0)=\\&\quad(\partial_1\partial_1 z+\mathrm{i}z\partial_1 A_1+2\mathrm{i}A_1\partial_1 z-A_1^2 z,\partial_1\partial_1\phi_3)=\\&\quad(\partial_1\partial_1(\phi_1+\mathrm{i}\phi_2)+\mathrm{i}(\phi_1+\mathrm{i}\phi_2)\partial_1 A_1+2\mathrm{i}A_1\partial_1(\phi_1+\mathrm{i}\phi_2)-A_1^2(\phi_1+\mathrm{i}\phi_2),\partial_1\partial_1\phi_3)=\\&\quad(\partial_1\partial_1\phi_1-\phi_2\partial_1 A_1-2A_1\partial_1\phi_2-A_1^2\phi_1,\partial_1\partial_1\phi_2+\phi_1\partial_1 A_1+2A_1\partial_1\phi_1-A_1^2\phi_2,\partial_1\partial_1\phi_3)\equiv\\&\quad(a,b,c).\end{aligned}$$

由上式可得:

$$\boldsymbol{\phi}\times D_1D_1\boldsymbol{\phi}=(c\phi_2-b\phi_3,a\phi_3-c\phi_1,b\phi_1-a\phi_2)=((c\phi_2-b\phi_3)+\mathrm{i}(a\phi_3-c\phi_1),b\phi_1-a\phi_2).$$

下面按上述方法计算 $\widetilde{\boldsymbol{\phi}}\times D_1D_1\widetilde{\boldsymbol{\phi}}$.

$$\begin{aligned}&D_1D_1\widetilde{\boldsymbol{\phi}}=((\partial_1\partial_1 z+\mathrm{i}z\partial_1 A_1+2\mathrm{i}A_1\partial_1 z-A_1^2 z)\mathrm{e}^{\mathrm{i}\chi},\partial_1\partial_1\phi_3)=\\&\quad((\partial_1\partial_1\phi_1-\phi_2\partial_1 A_1-2A_1\partial_1\phi_2-A_1^2\phi_1)\mathrm{e}^{\mathrm{i}\chi},(\partial_1\partial_1\phi_2+\phi_1\partial_1 A_1+2A_1\partial_1\phi_1-A_1^2\phi_2)\mathrm{e}^{\mathrm{i}\chi},\partial_1\partial_1\phi_3)=\\&\quad((a+\mathrm{i}b)\mathrm{e}^{\mathrm{i}\chi},\partial_1\partial_1\phi_3)=(a\cos\chi-b\sin\chi,a\sin\chi+b\cos\chi,\partial_1\partial_1\phi_3)\equiv(\tilde{a},\tilde{b},\tilde{c}).\end{aligned}$$

由上式可得 $\widetilde{\boldsymbol{\phi}}\times D_1D_1\widetilde{\boldsymbol{\phi}}=(\tilde{c}\tilde{\phi}_2-\tilde{b}\tilde{\phi}_3,\tilde{a}\tilde{\phi}_3-\tilde{c}\tilde{\phi}_1,\tilde{b}\tilde{\phi}_1-\tilde{a}\tilde{\phi}_2)$, 其中:

$$\tilde{c}\tilde{\phi}_2-\tilde{b}\tilde{\phi}_3+\mathrm{i}(\tilde{a}\tilde{\phi}_3-\tilde{c}\tilde{\phi}_1)=(\phi_1\sin\chi+\phi_2\cos\chi)c-$$
$$\phi_3(a\sin\chi+b\cos\chi)+\mathrm{i}\phi_3(a\cos\chi-b\sin\chi)-\mathrm{i}(\phi_1\cos\chi-\phi_2\sin\chi)c=$$
$$-\mathrm{i}c\phi_1(\cos\chi+\mathrm{i}\sin\chi)+c\phi_2(\cos\chi+\mathrm{i}\sin\chi)+\mathrm{i}a\phi_3(\cos\chi+\mathrm{i}\sin\chi)-b\phi_3(\cos\chi+\mathrm{i}\sin\chi)=$$
$$(c\phi_2-b\phi_3)(\cos\chi+\mathrm{i}\sin\chi)+\mathrm{i}(a\phi_3-c\phi_1)(\cos\chi+\mathrm{i}\sin\chi)=$$
$$(c\phi_2-b\phi_3)\mathrm{e}^{\mathrm{i}\chi}+\mathrm{i}(a\phi_3-c\phi_1)\mathrm{e}^{\mathrm{i}\chi},$$
$$\tilde{b}\tilde{\phi}_1-\tilde{a}\tilde{\phi}_2=(a\sin\chi+b\cos\chi)(\phi_1\cos\chi-\phi_2\sin\chi)-$$
$$(a\cos\chi-b\sin\chi)(\phi_1\sin\chi+\phi_2\cos\chi)=b\phi_1-a\phi_2.$$

由以上可得 $\tilde{\boldsymbol{\phi}}\times D_1D_1\tilde{\boldsymbol{\phi}}=(((\phi_2c-b\phi_3)+\mathrm{i}(a\phi_3-c\phi_1))\mathrm{e}^{\mathrm{i}\chi},b\phi_1-a\phi_2)$. 由此知,方程(1)在规范变化下保持不变.

再根据 $F_{\mu\nu}$ 的定义可知 $F_{01}=\partial_0A_1-\partial_1A_0$. 于是由函数 $\chi\in C^{\infty}(\mathbf{R}^{1+1})$ 知 F_{01} 经过变换后有

$$\tilde{F}_{01}=\partial_0\tilde{A}_1-\partial_1\tilde{A}_0=\partial_0(A_1-\partial_1\chi)-\partial_1(A_0-\partial_0\chi)=\partial_0A_1-\partial_1A_0.$$

又因为 $N|\boldsymbol{n}\times\boldsymbol{\phi}|^2=N|\mathrm{i}z|^2=N(\phi_1^2+\phi_2^2)=\tilde{N}|\mathrm{i}z\mathrm{e}^{\mathrm{i}\chi}|^2=\tilde{N}|\boldsymbol{n}\times\tilde{\boldsymbol{\phi}}|^2$,因此方程(2)满足规范不变性. 将 $D_1\boldsymbol{\phi}=(\partial_1z+\mathrm{i}A_1z,\partial_1\phi_3)$ 代入方程(3)中化简可得:

$$\partial_0N=-\langle\boldsymbol{n}\times\boldsymbol{\phi},D_1\boldsymbol{\phi}\rangle=-\langle(\mathrm{i}z,0),(\partial_1z+\mathrm{i}A_1z,\partial_1\phi_3)\rangle=-\mathrm{i}z\partial_1z+A_1|z|^2.$$

对上式进行变换后可得:

$$\partial_0\tilde{N}=-\langle\boldsymbol{n}\times\tilde{\boldsymbol{\phi}},D_1\tilde{\boldsymbol{\phi}}\rangle=-\langle(\mathrm{i}z\mathrm{e}^{\mathrm{i}\chi},0),(\partial_1z\mathrm{e}^{\mathrm{i}\chi}+\mathrm{i}A_1z\mathrm{e}^{\mathrm{i}\chi},\partial_1\phi_3)\rangle=-\mathrm{i}z\partial_1z+A_1|z|^2.$$

由上式可知方程(3)满足规范不变性. 由上述显然知方程(4)也满足规范不变性,证毕.

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