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一类双参数奇摄动方程非线性多点边值问题

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摘要: 讨论了一类具非线性多点边值条件的三阶微分方程的双参数奇摄动问题. 首先, 利用奇摄动方法求出问题的外部解; 然后, 引入两个不同的伸展变量构造了问题在边界附近的边界层校正项, 得到了所提问题的形式渐近解; 最后, 运用微分不等式理论证明了问题解的存在性及所得形式渐近解的一致有效性, 并用例子证明了该结果.

关键词: 奇摄动; 双参数; 非线性多点边值条件; 形式渐近解

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A class of nonlinear multi-point boundary value problems with two-parameter singularly perturbed equation

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Abstract: A class of singularly perturbed problems with two parameters for third-order differential equations with nonlinear multi-point boundary value conditions were discussed. Firstly, the outer solution was constructed by means of the singular perturbation method; Then, two different stretching variables were introduced, the boundary layer correction of solution were obtained, and the asymptotic analytic expansion solution to the original problem was also given; Finally, according to the theory of differential inequalities, the existence of solutions and the uniform validity of the asymptotic solutions were proved, and the result were proved by using an example.

Keywords: singular perturbation; two parameters; nonlinear multi-point boundary value conditions; formal asymptotic solutions

0 引言

近年来, 奇摄动边值问题的研究受到许多学者的关注, 并得到许多研究成果^[1-7]; 但其中大部分的研究结果是关于两点或三点边值条件的奇摄动问题, 而对于多点边值条件的奇摄动问题研究得较少. 文献[8]的作者用 Liouville-green 变换得到了奇摄动二阶微分方程多点边值问题的渐近解; 文献[9]的作者利用微分不等式理论和 Leray-Schauder 度理论研究了一类三阶微分方程的多点边值条件的奇摄动问

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题,并得到了问题解的存在唯一性和渐近估计结果;文献[10]的作者在文献[9]的研究基础上将线性多点边值条件推广到非线性多点边值条件,研究了带有非线性多点边值条件的三阶微分方程的奇摄动问题.但目前为止,对于含双参数带有非线性多点边值条件的三阶微分方程的奇摄动问题的研究尚未见有文献报道;为此,本文在文献[10]的研究基础上,考虑如下类带有非线性多点边值条件的三阶微分方程的双参数奇摄动问题:

$$\varepsilon x'''(t) + f(t, x(t), x'(t), \mu x''(t)) = 0, 0 \leq t \leq 1; \quad (1)$$

$$x(0) = 0; \quad (2)$$

$$g(x'(0), x''(0), x(\xi_1), x(\xi_2), \dots, x(\xi_{m-2})) = A; \quad (3)$$

$$h(x'(1), x''(1), x(\eta_1), x(\eta_2), \dots, x(\eta_{n-2})) = B. \quad (4)$$

其中 ε 和 μ 均是正的小参数, $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$, $0 < \eta_1 < \eta_2 < \dots < \eta_{n-2} < 1$, A 和 B 为常数. 现作如下假设:

$$[H_1] \quad \text{当 } \mu \rightarrow 0 \text{ 时, } \frac{\varepsilon}{\mu^2} \rightarrow 0.$$

$[H_2]$ 问题(1)–(4)的退化问题 $f(t, X_{0,0}, X'_{0,0}, 0) = 0$, $X_{0,0}(0) = 0$ 在 $t \in [0, 1]$ 上存在充分光滑的解 $X_{0,0} = X_{0,0}(t)$.

$[H_3]$ 函数 $f(t, x, y, z)$, $g(x_1, x_2, \dots, x_m)$, $h(y_1, y_2, \dots, y_n)$ 的变元在其相应的区域内充分光滑, 且存在正常数 $l_0, L (L \geq 2l_0)$, l_1, l_2, k 使得 $0 \leq f_x \leq l_0$, $f_y \leq -L$, $f_z \leq -k$, $g_{x_1} + \sum_{i_1=1}^{m-2} g_{x_{i_1+2}} \xi_{i_1} \geq l_1$,

$$h_{y_1} + \sum_{i_2=1}^{n-2} h_{y_{i_2+2}} \eta_{i_2} \geq l_2, \quad g_{x_{i_3}} < 0 (i_3 = 2, \dots, m), \quad h_{y_2} > 0, \quad h_{y_{i_4}} < 0 (i_4 = 3, \dots, n).$$

$[H_4]$ 由方程 $g(X'_{0,0}(0), X''_{0,0}(0) + U_1, X_{0,0}(\xi_1), \dots, X_{0,0}(\xi_{m-2})) = A$ 可求出 U_1 , 由方程 $h(X'_{0,0}(1), X''_{0,0}(1) + U_2, X_{0,0}(\eta_1), \dots, X_{0,0}(\eta_{n-2})) = B$ 可求出 U_2 .

1 外部解的构造

令 $\xi = \mu$, $\eta = \frac{\varepsilon}{\mu^2}$, 则方程(1)可转化为

$$\xi^2 \eta x'''(t) + f(t, x(t), x'(t), \xi x''(t)) = 0. \quad (5)$$

设问题(1)–(4)的外部解的形式渐近式为

$$X(t, \xi, \eta) \sim \sum_{i,j=0}^{\infty} X_{i,j}(t) \xi^i \eta^j. \quad (6)$$

将式(6)代入式(5)可得:

$$f(t, X_{0,0}, X'_{0,0}, 0) = 0; \quad (7)$$

$$f_x(t, X_{0,0}, X'_{0,0}, 0) X_{i,j} + f_y(t, X_{0,0}, X'_{0,0}, 0) X'_{i,j} = F_{i,j}(t), \quad i+j \geq 1, \quad (8)$$

其中 $F_{i,j}(t)$ 是由 $X_{s,q}, X'_{s,q}, X''_{s,q}, X'''_{s,q} (s+q < i+j)$ 依次确定的函数. 由上述可知, 式(7)、(8)即为问题(1)–(4)的外部解的递推方程.

2 边界层校正项

由假设可知, 问题(1)–(4)在 $t=0$ 和 $t=1$ 附近处各有一个边界层. 首先, 在 $t=0$ 处构造边界层的校正项, 同时引进伸展变量 $\tau_1 (\tau_1 = \frac{t}{\xi})$, 并令

$$x(t, \xi, \eta) = X(t, \xi, \eta) + \xi^2 W(\tau_1, \xi, \eta), \quad (9)$$

其中

$$W(\tau_1, \xi, \eta) \sim \sum_{i,j=0}^{\infty} W_{i,j}(\tau_1) \xi^i \eta^j, \quad (10)$$

且 $W(\tau_1, \xi, \eta)$ 具有如下性质: $\lim_{\tau_1 \rightarrow +\infty} W(\tau_1, \xi, \eta) = 0$. 将式(6)、(9)、(10) 代入式(5) 可得:

$$f_y(0, X_{0,0}(0), X'_{0,0}(0), 0) \frac{dW_{0,0}}{d\tau_1} + f_z(0, X_{0,0}(0), X'_{0,0}(0), 0) \frac{d^2 W_{0,0}}{d\tau_1^2} = 0; \quad (11)$$

$$f_y(0, X_{0,0}(0), X'_{0,0}(0), 0) \frac{dW_{i,j}}{d\tau_1} + f_z(0, X_{0,0}(0), X'_{0,0}(0), 0) \frac{d^2 W_{i,j}}{d\tau_1^2} = \tilde{F}_{i,j}(\tau_1), i+j \geq 1, \quad (12)$$

其中 $\tilde{F}_{i,j}(\tau_1)$ 是关于 $\tau_1, W_{s,q}(s+q < i+j)$ 及其各阶导数的多项式函数.

类似地, 在 $t=1$ 处构造边界层的校正项, 并令

$$x(t, \xi, \eta) = X(t, \xi, \eta) + \xi^2 \eta^2 Q(\tau_2, \xi, \eta), \quad (13)$$

其中 $\tau_2 = \frac{1-t}{\xi\eta}$ 为伸长变量,

$$Q(\tau_2, \xi, \eta) \sim \sum_{i,j=0}^{\infty} Q_{i,j}(\tau_2) \xi^i \eta^j, \quad (14)$$

且 $Q(\tau_2, \xi, \eta)$ 具有如下性质: $\lim_{\tau_2 \rightarrow +\infty} Q(\tau_2, \xi, \eta) = \lim_{\tau_2 \rightarrow +\infty} \frac{dQ}{d\tau_2}(\tau_2, \xi, \eta) = 0$. 将式(6)、(13)、(14) 代入式(5), 则有:

$$\frac{d^3 Q_{0,0}}{d\tau_2^3} = f_z(1, X_{0,0}(1), X'_{0,0}(1), 0) \frac{d^2 Q_{0,0}}{d\tau_2^2}; \quad (15)$$

$$\frac{d^3 Q_{i,j}}{d\tau_2^3} = f_z(1, X_{0,0}(1), X'_{0,0}(1), 0) \frac{d^2 Q_{i,j}}{d\tau_2^2} + \bar{F}_{i,j}(\tau_2), i+j \geq 1, \quad (16)$$

其中 $\bar{F}_{i,j}(\tau_2)$ 是关于 $\tau_2, Q_{s,q}(s+q < i+j)$ 及其各阶导数的多项式函数.

为确定 $X_{i,j}(t), W_{i,j}(\tau_1), Q_{i,j}(\tau_2)$ 的定解条件, 将 $x(t, \xi, \eta) = X(t, \xi, \eta) + \xi^2 W(\tau_1, \xi, \eta) + \xi^2 \eta^2 Q(\tau_2, \xi, \eta)$ 代入式(2)–(4) 得:

$$X_{i,j}(0) = 0, i < 2; \quad (17)$$

$$X_{i,j}(0) = -W_{i-2,j}(0), i \geq 2; \quad (18)$$

$$g(X'_{0,0}(0), X''_{0,0}(0) + \frac{d^2 W_{0,0}}{d\tau_1^2} |_{\tau_1=0}, X_{0,0}(\xi_1), \dots, X_{0,0}(\xi_{m-2})) = A; \quad (19)$$

$$g_{x_2}(X'_{0,0}(0), X''_{0,0}(0) + \frac{d^2 W_{0,0}}{d\tau_1^2} |_{\tau_1=0}, X_{0,0}(\xi_1), \dots, X_{0,0}(\xi_{m-2})) \frac{d^2 W_{i,j}}{d\tau_1^2} |_{\tau_1=0} = G_{i,j}, i+j \geq 1; \quad (20)$$

$$h(X'_{0,0}(1), X''_{0,0}(1) + \frac{d^2 Q_{0,0}}{d\tau_2^2} |_{\tau_2=0}, X_{0,0}(\eta_1), \dots, X_{0,0}(\eta_{n-2})) = B; \quad (21)$$

$$h_{y_2}(X'_{0,0}(1), X''_{0,0}(1) + \frac{d^2 Q_{0,0}}{d\tau_2^2} |_{\tau_2=0}, X_{0,0}(\eta_1), \dots, X_{0,0}(\eta_{n-2})) \frac{d^2 Q_{i,j}}{d\tau_2^2} |_{\tau_2=0} = H_{i,j}, i+j \geq 1. \quad (22)$$

其中 $G_{i,j}$ 和 $H_{i,j}$ 是确定的数值. 由假设 $[H_4]$ 及式(19) 和(21) 可得: $\frac{d^2 W_{0,0}}{d\tau_1^2} |_{\tau_1=0}$, 记作 A_1 ; $\frac{d^2 Q_{0,0}}{d\tau_2^2} |_{\tau_2=0}$, 记作 A_2 . 由于 $X_{0,0}(t)$ 满足定解问题 $f(t, X_{0,0}, X'_{0,0}, 0) = 0 (0 \leq t \leq 1)$, $X_{0,0}(0) = 0$, 因此由假设 $[H_2]$ 可知该定解问题的解存在, 为 $X_{0,0} = X_{0,0}(t)$.

根据递推方程(8)、(11)、(12)、(15)、(16) 和定解条件(17)–(22) 以及 $W(\tau_1, \xi, \eta)$ 和 $Q(\tau_2, \xi, \eta)$

的性质,运用交替迭代的方法可依次求出 $X_{i,j}(t), W_{i,j}(\tau_1), Q_{i,j}(\tau_2)$, 并且有 $W_{0,0}(\tau_1) = \frac{A_1}{\lambda_1^2} e^{\lambda_1 \tau_1}$, $Q_{0,0}(\tau_2) = \frac{A_2}{\lambda_2^2} e^{\lambda_2 \tau_2}$. 式中 λ_1 为式(11)的特征方程的负根, λ_2 为式(15)的特征方程的负根. 由 $W_{0,0}(\tau_1)$ 和 $Q_{0,0}(\tau_2)$ 以及 $\tilde{F}_{i,j}(\tau_1)$ 和 $\bar{F}_{i,j}(\tau_2)$ 的构造可知, $W_{i,j}(\tau_1)$ 和 $Q_{i,j}(\tau_2)$ 都具有指数型衰减的特征.

引进光滑函数 $\varphi(t) \in C^\infty[0,1]$, 使得 $\varphi(t) = \begin{cases} 1, & t \in [0, \sigma]; \\ p(t), & t \in [\sigma, 1-\sigma]; \\ 0, & t \in [1-\sigma, 1], \end{cases}$ 其中 $p(t)$ 是一个多项式函数, σ 为 $0 < \sigma < 1$ 的常数. 根据假设 $[H_1]$, 令

$$x_N(t, \xi, \eta) = \sum_{i+j=0}^N X_{i,j}(t) \xi^i \eta^j + \varphi(t) \xi^2 \sum_{i+j=0}^N W_{i,j}\left(\frac{t}{\xi}\right) \xi^i \eta^j + \varphi(1-t) \xi^2 \eta^2 \sum_{i+j=0}^N Q_{i,j}\left(\frac{1-t}{\xi \eta}\right) \xi^i \eta^j, \quad (23)$$

由此即可得问题(1)–(4)的 N 阶形式渐近解.

3 结果及其证明

定义 1^[11] 若函数 $u(t), v(t) \in C^3[0,1]$ 满足

$$\begin{aligned} u'''(t) + f(t, u(t), u'(t), u''(t)) &\geq 0, \\ u(0) &= 0, \\ g(u'(0), u''(0), u(\xi_1), u(\xi_2), \dots, u(\xi_{m-2})) &\leq A, \\ h(u'(1), u''(1), u(\eta_1), u(\eta_2), \dots, u(\eta_{n-2})) &\leq B, \\ v'''(t) + f(t, v(t), v'(t), v''(t)) &\leq 0, \\ v(0) &= 0, \\ g(v'(0), v''(0), v(\xi_1), v(\xi_2), \dots, v(\xi_{m-2})) &\geq A, \\ h(v'(1), v''(1), v(\eta_1), v(\eta_2), \dots, v(\eta_{n-2})) &\geq B, \end{aligned}$$

则称 $u(t)$ 和 $v(t)$ 分别是如下边值问题的下解和上解:

$$x'''(t) + f(t, x(t), x'(t), x''(t)) = 0, \quad 0 < t < 1; \quad (24)$$

$$x(0) = 0; \quad (25)$$

$$g(x'(0), x''(0), x(\xi_1), x(\xi_2), \dots, x(\xi_{m-2})) = A; \quad (26)$$

$$h(x'(1), x''(1), x(\eta_1), x(\eta_2), \dots, x(\eta_{n-2})) = B. \quad (27)$$

定义 2^[11] 设 D 为 $[0,1] \times \mathbf{R}^3$ 的子集, 且给定 $a > 0$. 若存在一个正函数 $\Phi: [0, +\infty) \rightarrow (a, +\infty)$

使得 $|f(x, y, z)| \leq \Phi(|z|)$, $\int_0^{+\infty} \frac{s}{\Phi(s)} ds = +\infty$, 则称连续函数 $f: D \rightarrow \mathbf{R}$ 在 D 上满足 Nagumo 条件.

引理 1^[11] 若边值问题(24)–(27) 满足如下条件:

(A₁) 存在下解 $u(t)$ 和上解 $v(t)$, 且当 $t \in [0,1]$ 时, 有 $u'(t) \leq v'(t)$;

(A₂) 函数 $f(t, x, y, z)$ 在 $[0,1] \times [u(t), v(t)] \times \mathbf{R}^2$ 上连续且关于 x 递增, 并且 $f(t, x, y, z)$ 在 $[0,1] \times [u(t), v(t)] \times [u'(t), v'(t)] \times \mathbf{R}$ 上满足 Nagumo 条件;

(A₃) 若函数 $g(x_1, x_2, \dots, x_m)$ 在 \mathbf{R}^m 上连续, 且关于 x_2, \dots, x_m 递减; 若函数 $h(y_1, y_2, \dots, y_n)$ 在 \mathbf{R}^n 上连续, 且关于 y_2 递增, 关于 y_3, \dots, y_n 递减;

则边值问题(24)–(27) 至少存在一个解 $x(t) \in C^3[0,1]$, 使得 $u(t) \leq x(t) \leq v(t)$, $u'(t) \leq x'(t) \leq v'(t)$, $t \in [0,1]$.

定理 1 若假设 $[H_1]$ – $[H_4]$ 成立, 且 $f(t, x, y, z)$ 在 $[0,1] \times \mathbf{R}^3$ 上满足 Nagumo 条件, 则问题(1)–(4) 存在解 $x(t, \xi, \eta) \in C^3[0,1]$, 且 $x(t, \xi, \eta)$ 满足 $x(t, \xi, \eta) - x_N(t, \xi, \eta) = O(\rho^{N+1})$, $x'(t, \xi,$

$\eta) - x'_N(t, \xi, \eta) = O(\rho^{N+1})$, 其中 $\rho = \max\{\xi, \eta\}$, $\xi = \mu$, $\eta = \frac{\varepsilon}{\mu^2}$, x_N 由式(23) 给出.

证明 构造辅助函数 $u(t, \xi, \eta) = x_N(t, \xi, \eta) - rt\rho^{N+1}$, $v(t, \xi, \eta) = x_N(t, \xi, \eta) + rt\rho^{N+1}$, 其中 r 为待定的充分大的正常数. 由该函数显然可得: $u(t, \xi, \eta) \leq v(t, \xi, \eta)$, $t \in [0, 1]$; $u'(t, \xi, \eta) \leq v'(t, \xi, \eta)$, $t \in [0, 1]$; $u(0, \xi, \eta) = v(0, \xi, \eta) = 0$. 另外, 由微分中值定理可知, 存在正常数 C_1 和 C_2 , 使得:

$$\begin{aligned} & g(u'(0, \xi, \eta), u''(0, \xi, \eta), u(\xi_1, \xi, \eta), u(\xi_2, \xi, \eta), \dots, u(\xi_{m-2}, \xi, \eta)) \leq \\ & g(x'_N(0, \xi, \eta), x''_N(0, \xi, \eta), x_N(\xi_1, \xi, \eta), x_N(\xi_2, \xi, \eta), \dots, x_N(\xi_{m-2}, \xi, \eta) - l_1 r \rho^{N+1}) \leq \\ & g(X'_{0,0}(0), X''_{0,0}(0) + \frac{d^2 W_{0,0}}{d\tau_1^2} \Big|_{\tau_1=0}, X_{0,0}(\xi_1), X_{0,0}(\xi_2), \dots, X_{0,0}(\xi_{m-2})) + \\ & \sum_{i+j=1}^N \left[g_{x_2}(X'_{0,0}(0), X''_{0,0}(0) + \frac{d^2 W_{0,0}}{d\tau_1^2} \Big|_{\tau_1=0}, X_{0,0}(\xi_1), \dots, X_{0,0}(\xi_{m-2})) \frac{d^2 W_{i,j}}{d\tau_1^2} \Big|_{\tau_1=0} - G_{i,j} \right] \xi^i \eta^j + \\ & C_1 \rho^{N+1} - l_1 r \rho^{N+1} = A + (C_1 - l_1 r) \rho^{N+1}, \\ & h(u'(1, \xi, \eta), u''(1, \xi, \eta), u(\eta_1, \xi, \eta), u(\eta_2, \xi, \eta), \dots, u(\eta_{n-2}, \xi, \eta)) \leq \\ & h(x'_N(1, \xi, \eta), x''_N(1, \xi, \eta), x_N(\eta_1, \xi, \eta), x_N(\eta_2, \xi, \eta), \dots, x_N(\eta_{n-2}, \xi, \eta) - l_2 r \rho^{N+1}) \leq \\ & h(X'_{0,0}(1), X''_{0,0}(1) + \frac{d^2 Q_{0,0}}{d\tau_2^2} \Big|_{\tau_2=0}, X_{0,0}(\eta_1), X_{0,0}(\eta_2), \dots, X_{0,0}(\eta_{n-2})) + \\ & \sum_{i+j=1}^N \left[h_{y_2}(X'_{0,0}(1), X''_{0,0}(1) + \frac{d^2 Q_{0,0}}{d\tau_2^2} \Big|_{\tau_2=0}, X_{0,0}(\eta_1), \dots, X_{0,0}(\eta_{n-2})) \frac{d^2 Q_{i,j}}{d\tau_2^2} \Big|_{\tau_2=0} - H_{i,j} \right] \xi^i \eta^j + \\ & C_2 \rho^{N+1} - l_2 r \rho^{N+1} = B + (C_2 - l_2 r) \rho^{N+1}. \end{aligned}$$

由上述可知只需 $r \geq \max\left\{\frac{C_1}{l_1}, \frac{C_2}{l_2}\right\}$ 即可得:

$$\begin{aligned} & g(u'(0, \xi, \eta), u''(0, \xi, \eta), u(\xi_1, \xi, \eta), u(\xi_2, \xi, \eta), \dots, u(\xi_{m-2}, \xi, \eta)) \leq A, \\ & h(u'(1, \xi, \eta), u''(1, \xi, \eta), u(\eta_1, \xi, \eta), u(\eta_2, \xi, \eta), \dots, u(\eta_{n-2}, \xi, \eta)) \leq B. \end{aligned}$$

类似地, 只需 $r \geq \max\left\{\frac{C_1}{l_1}, \frac{C_2}{l_2}\right\}$ 即可得:

$$\begin{aligned} & g(v'(0, \xi, \eta), v''(0, \xi, \eta), v(\xi_1, \xi, \eta), v(\xi_2, \xi, \eta), \dots, v(\xi_{m-2}, \xi, \eta)) \geq A, \\ & h(v'(1, \xi, \eta), v''(1, \xi, \eta), v(\eta_1, \xi, \eta), v(\eta_2, \xi, \eta), \dots, v(\eta_{n-2}, \xi, \eta)) \geq B. \end{aligned}$$

下面证明:

$$\begin{aligned} & \varepsilon u'''(t, \xi, \eta) + f(t, u(t, \xi, \eta), u'(t, \xi, \eta), \mu u''(t, \xi, \eta)) \geq 0, \quad 0 < t < 1; \\ & \varepsilon v'''(t, \xi, \eta) + f(t, v(t, \xi, \eta), v'(t, \xi, \eta), \mu v''(t, \xi, \eta)) \leq 0, \quad 0 < t < 1. \\ & \varepsilon u'''(t, \xi, \eta) + f(t, u(t, \xi, \eta), u'(t, \xi, \eta), \mu u''(t, \xi, \eta)) = \varepsilon x_N''' + f(t, x_N, x'_N, \mu x_N'') - f_x(t, \theta_0, \theta_1, \\ & \mu x_N'') r t \rho^{N+1} - f_y(t, \theta_0, \theta_1, \mu x_N'') r \rho^{N+1} \geq \varepsilon x_N''' + f(t, x_N, x'_N, \mu x_N'') + l_0 r \rho^{N+1}, \text{ 其中 } \theta_0 \in (u, x_N), \theta_1 \in \\ & (u', x'_N). \text{ 当 } x \in [0, \sigma] \text{ 时, 由外部解和左边界层的构造可知, 存在正常数 } C_3 \text{ 和 } C_4, \text{ 使得} \end{aligned}$$

$$\begin{aligned} & \varepsilon x_N''' + f(t, x_N, x'_N, \mu x_N'') + l_0 r \rho^{N+1} \geq f(t, X_{0,0}, X'_{0,0}, 0) + \\ & \sum_{i+j=1}^N [f_x(t, X_{0,0}, X'_{0,0}, 0) X_{i,j} + f_y(t, X_{0,0}, X'_{0,0}, 0) X'_{i,j} - F_{i,j}(t)] \xi^i \eta^j - \\ & C_3 \rho^{N+1} + \left[f_y(0, X_{0,0}(0), X'_{0,0}(0), 0) \frac{dW_{0,0}}{d\tau_1} + f_z(0, X_{0,0}(0), X'_{0,0}(0), 0) \frac{d^2 W_{0,0}}{d\tau_1^2} \right] \xi^1 \eta^0 + \\ & \sum_{i+j=1}^N \left[f_y(0, X_{0,0}(0), X'_{0,0}(0), 0) \frac{dW_{i,j}}{d\tau_1} + f_z(0, X_{0,0}(0), X'_{0,0}(0), 0) \frac{d^2 W_{i,j}}{d\tau_1^2} - \tilde{F}_{i,j}(\tau_1) \right] \xi^{i+1} \eta^j - \\ & C_4 \rho^{N+1} + l_0 r \rho^{N+1} = (l_0 r - C_3 - C_4) \rho^{N+1}. \end{aligned}$$

类似地, 当 $x \in [1 - \sigma, 1]$ 时, 由外部解和右边界层的构造可知, 存在正常数 C_5 , 使得

$$\varepsilon x_N''' + f(t, x_N, x_N', \mu x_N'') + l_0 r \rho^{N+1} \geq (l_0 r - C_3 - C_5) \rho^{N+1}.$$

当 $x \in [\sigma, 1 - \sigma]$ 时, 由 $W_{i,j}(\tau_1), Q_{i,j}(\tau_2)$ 的边界层性态可知, 存在正常数 C_6 , 使得

$$\begin{aligned} \varepsilon x_N''' + f(t, x_N, x_N', \mu x_N'') + l_0 r \rho^{N+1} \geq & \\ -C_3 \rho^{N+1} + \xi^2 \eta \left[\varphi(t) \xi^2 \sum_{i+j=0}^N W_{i,j} \left(\frac{t}{\xi} \right) \xi^i \eta^j + \varphi(1-t) \xi^2 \eta^2 \sum_{i+j=0}^N Q_{i,j} \left(\frac{1-t}{\xi \eta} \right) \xi^i \eta^j \right]^{(3)} + & \\ f_x(t, \zeta_1, \zeta_2, \zeta_3) (x_N - \sum_{i+j=0}^N X_{i,j}(t) \xi^i \eta^j) + f_y(t, \zeta_1, \zeta_2, \zeta_3) (x_N' - \sum_{i+j=0}^N X'_{i,j}(t) \xi^i \eta^j) + & \\ f_z(t, \zeta_1, \zeta_2, \zeta_3) (\xi x_N'' - \xi \sum_{i+j=0}^N X''_{i,j}(t) \xi^i \eta^j) + l_0 r \rho^{N+1} \geq (l_0 r - C_3 - C_6) \rho^{N+1}, & \end{aligned}$$

其中 ζ_1 在 x_N 与 $\sum_{i+j=0}^N X_{i,j}(t) \xi^i \eta^j$ 之间, ζ_2 在 x_N' 与 $\sum_{i+j=0}^N X'_{i,j}(t) \xi^i \eta^j$ 之间, ζ_3 在 $\xi x_N''$ 与 $\xi \sum_{i+j=0}^N X''_{i,j}(t) \xi^i \eta^j$ 之间.

由上述可知, 只要 $r \geq \max \left\{ \frac{C_3 + C_4}{l_0}, \frac{C_3 + C_5}{l_0}, \frac{C_3 + C_6}{l_0} \right\}$, 对一切 $t \in [0, 1]$ 有 $\varepsilon u'''(t, \xi, \eta) + f(t, u(t, \xi, \eta), u'(t, \xi, \eta), \mu u''(t, \xi, \eta)) \geq 0$. 类似可证, 当 $t \in [0, 1]$ 时, 能够找到充分大的 r , 使 $\varepsilon v'''(t, \xi, \eta) + f(t, v(t, \xi, \eta), v'(t, \xi, \eta), \mu v''(t, \xi, \eta)) \leq 0$. 再由引理 1 可知, 只要 r 充分大, 边值问题(1)–(4)存在解 $x(t, \xi, \eta) \in C^3[0, 1]$, 且满足 $x(t, \xi, \eta) - x_N(t, \xi, \eta) = O(\rho^{N+1})$, $x'(t, \xi, \eta) - x_N'(t, \xi, \eta) = O(\rho^{N+1})$. 定理证毕.

4 算例

考虑满足假设条件 $[H_1]$ – $[H_4]$ 的如下混合边值条件的三阶微分方程的奇摄动问题:

$$\varepsilon x''' - \mu x'' - 2x' + x + t = 0, \quad 0 < t < 1, \quad 0 < \varepsilon \ll 1, \quad 0 < \mu \ll 1; \quad (28)$$

$$x(0) = 0; \quad (29)$$

$$4x'(0) - x''(0) - \frac{1}{2}x\left(\frac{1}{3}\right) = -e^{\frac{1}{6}}; \quad (30)$$

$$x'(1) + 2x''(1) - \frac{1}{2}x\left(\frac{1}{2}\right) = 2e^{\frac{1}{2}} - e^{\frac{1}{4}} + 1. \quad (31)$$

由假设 $[H_2]$ 可知, 问题(28)–(31)的退化问题为: $-2X'_{0,0} + X_{0,0} + t = 0, X_{0,0}(0) = 0$. 在 $t \in [0, 1]$ 上, 该退化问题的解为 $X_{0,0}(t) = 2e^{\frac{1}{2}t} - t - 2$. 在 $x = 0$ 处构造边界层的校正项, 于是有 $\frac{d^2 W_{0,0}}{d\tau_1^2} + 2 \frac{dW_{0,0}}{d\tau_1} = 0, \frac{d^2 W_{0,0}}{d\tau_1^2} \big|_{\tau_1=0} = \frac{2}{3}$. 在 $x = 1$ 处构造边界层的校正项, 于是有 $\frac{d^3 Q_{0,0}}{d\tau_2^3} = -\frac{d^2 Q_{0,0}}{d\tau_2^2}, \frac{d^2 Q_{0,0}}{d\tau_2^2} \big|_{\tau_2=0} = \frac{3}{8}$. 由 $W(\tau_1, \xi, \eta)$ 和 $Q(\tau_2, \xi, \eta)$ 的性质及上式可得 $W_{0,0}(\tau_1) = \frac{1}{6}e^{-2\tau_1}, Q_{0,0}(\tau_2) = \frac{3}{8}e^{-\tau_2}$. 再由假设 $[H_1]$ 和问题(28)–(31)形式渐近解的构造可得问题(28)–(31)的零阶形式渐近解为:

$$x_0(t, \xi, \eta) = 2e^{\frac{1}{2}t} - t - 2 + \frac{1}{6}\mu^2 \varphi(t)e^{-2\tau_1} + \frac{3}{8}\frac{\varepsilon^2}{\mu^2} \varphi(1-t)e^{-\tau_2}.$$

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