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一类带有 p -Laplacian 算子与积分边值条件的 Caputo 分数阶 q -差分方程解的存在性

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摘要: 研究了一类带有 p -Laplacian 算子与积分边界条件的 Caputo 分数阶 q -差分方程:

$$\begin{cases} {}^C D_q^\beta(\phi_p({}^C D_q^\alpha u(t))) + f(t, u(t)) = 0, t \in [0, 1]; \\ u(1) = \lambda \int_0^1 u(s) d_q s, D_q u(0) = 0, \\ {}^C D_q^\alpha u(1) = b {}^C D_q^\alpha u(\xi). \end{cases}$$

首先利用 Arzelà-Ascoli 定理与 Schauder 不动点定理证明了此类 Caputo 分数阶 q -差分方程解的存在性, 然后利用一个实例验证了文中所得的主要结论.

关键词: p -Laplacian 算子; q -差分方程; Caputo 分数阶导数; Arzelà-Ascoli 定理; Schauder 不动点定理

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Existence of solutions to a class of Caputo fractional q -difference equations with p -Laplacian operator and integral boundary value conditions

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Abstract: A class of Caputo fractional q -difference equations with p -Laplacian operator and integral boundary

conditions are studied $\begin{cases} {}^C D_q^\beta(\phi_p({}^C D_q^\alpha u(t))) + f(t, u(t)) = 0, t \in [0, 1]; \\ u(1) = \lambda \int_0^1 u(s) d_q s, D_q u(0) = 0, \\ {}^C D_q^\alpha u(1) = b {}^C D_q^\alpha u(\xi). \end{cases}$ First, the Arzelà-Ascoli theorem

and the Schauder fixed point theorem are used to prove the relevant conclusions of the existence of such Caputo fractional q -difference equations, and then an example is used to verify the main conclusions obtained in the article.

Keywords: p -Laplacian operator; q -difference equation; Caputo fractional derivative; Arzelà-Ascoli theorem; Schauder fixed point theorem

0 引言

近年来许多学者研究了带有边值条件的分数阶微分方程与 q -差分方程, 并取得了许多较好的结果. 例如: 在文献[1]中, Yan 等利用不动点定理研究了带有 p -Laplacian 算子的分数阶微分方程的边值

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问题:

$$\begin{cases} -D_{0+}^{\alpha}(\phi_p(D_{0+}^{\beta}u(t))) = f(t, u(t)), \\ u(0) = u(1) = u'(0) = u'(1), \\ D_{0+}^{\beta}u(0) = 0, D_{0+}^{\beta}u(1) = bD_{0+}^{\beta}u(\eta), \end{cases}$$

其中 $1 < \alpha \leq 2$, $3 < \beta \leq 4$, $0 < \eta < 1$, $0 < b < \eta^{(1-\alpha)(p-1)}$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p$, $\phi_p^{-1}(s) = \phi_q(s)$, D_{0+}^{α} 和 D_{0+}^{β} 是标准 Riemann-Liouville 导数, $f(t, u): (0, 1) \times (0, \infty) \rightarrow [0, \infty)$ 是给定的连续函数. 在文献 [2] 中, Xie 等研究了带有 p -Laplacian 算子和分数阶边界条件的非线性分数阶微分方程:

$$\begin{cases} {}^C D_{0+}^{\beta}(\phi_p({}^C D_{0+}^{\alpha}u(t))) + f(t, u(t)) = 0, t \in [0, 1]; \\ u(1) = \lambda \int_0^1 u(s) ds, u'(1) = 0; \\ {}^C D_{0+}^{\alpha}u(1) = b {}^C D_{0+}^{\alpha}u(\xi), \end{cases}$$

其中 $1 < \alpha \leq 2$, $0 < \beta \leq 1$, $0 < \xi, b, \lambda < 1$, ${}^C D_{0+}^{\alpha}$ 与 ${}^C D_{0+}^{\beta}$ 是 Caputo 分数阶导数. 在文献 [3] 中, Chen 等研究了带有边值的 q -差分方程:

$$\begin{cases} ({}^C D_q^v x)(t) + f(t, x(t)) = 0, 0 < q < 1, 0 < t < 1, 2 < v < 3; \\ x(0) = D_q^2 x(0) = 0; \\ x(1) = \lambda \int_0^1 x(s) d_q s, \end{cases}$$

其中 ${}^C D_q^v$ 是 v 阶 Caputo 分数阶 q -导数, $f \in C([0, 1] \times [0, +\infty)) \rightarrow [0, +\infty)$ 是连续函数.

基于上述文献研究, 本文研究如下一类带有 p -Laplacian 算子与积分边界条件的 Caputo 分数阶 q -差分方程:

$$\begin{cases} {}^C D_q^{\beta}(\phi_p({}^C D_q^{\alpha}u(t))) + f(t, u(t)) = 0, t \in [0, 1]; \\ u(1) = \lambda \int_0^1 u(s) d_q s, D_q u(0) = 0; \\ {}^C D_q^{\alpha}u(1) = b {}^C D_q^{\alpha}u(\xi). \end{cases} \quad (1)$$

其中 $1 < \alpha < 2$, $0 < \beta < 1$, $0 < \xi, b, \lambda < 1$, ${}^C D_q^{\alpha}$ 和 ${}^C D_q^{\beta}$ 分别为阶数为 α 和 β 的 Caputo 分数阶 q -导数. $\phi_p(s) = |s|^{p-2}s$ 为 p -Laplacian 算子, 且 $\frac{1}{p} + \frac{1}{q^*} = 1$, $p > 1$, $\phi_p^{-1}(s) = \phi_{q^*}(s)$; $f(t, u): [0, 1] \times [0, \infty) \rightarrow [0, \infty)$.

1 预备知识及其原理

令 $0 < q < 1$, 并定义 $[\alpha]_q = \frac{1-q^{\alpha}}{1-q}$, $\alpha \in \mathbf{R}$. 当 $n \in \mathbf{N}$ 时, q -幂函数为 $(t-s)^0 = 1$, $(t-s)^n = \prod_{k=0}^{n-1} (t-sq^k)$, $t, s \in \mathbf{R}$. 一般地, 若 $\gamma \in \mathbf{R}$, 则 $(t-s)^{(\gamma)} = t^{\gamma} \prod_{n=0}^{\infty} \frac{t-sq^n}{t-sq^{n+\gamma}}$. q -gamma 函数为 $\Gamma_q(s) = (1-q)^{(s-1)}(1-q)^{1-s}$, $s \in \mathbf{R} \setminus \{0, -1, -2, \dots\}$, 且满足 $\Gamma_q(s+1) = [s]_q \Gamma_q(s)$.

当 $0 < q < 1$ 时, 函数 f 的 q -导数为 $(D_q f)(t) = \frac{d_q}{d_q t} f(t) = \frac{f(t) - f(qt)}{(1-q)t}$, $(D_q f)(0) = \lim_{t \rightarrow 0} (D_q f)(t)$, $t \neq 0$. 高阶 q -导数为 $(D_q^0 f)(t) = f(t)$, $(D_q^n f)(t) = D_q(D_q^{n-1} f)(t)$, $n \in \mathbf{N}$. 若级数 $\sum_{n=0}^{\infty} f(tq^n)q^n$ 收敛, 则函数 f 的 q -积分可定义为 $(I_q f)(t) = \int_0^t f(s) d_q s = t(1-q) \sum_{n=0}^{\infty} f(tq^n)q^n$, $t \in [0, b]$; 函数 f 从 a 到 b 的 q -积分为 $\int_a^b f(s) d_q s = \int_0^b f(s) d_q s - \int_0^a f(s) d_q s$, $a \in [0, b]$; 函数 f 的高阶

q -积分为 $(I_q^0 f)(t) = f(t)$, $(I_q^n f)(t) = I_q(I_q^{n-1} f)(t)$, $n \in \mathbf{N}$.

定义 1^[4] 令 $v \geq 0$, $f \in L^1([0, 1])$, Riemann-Liouville 型分数阶 q -积分为 $(I_q^0 f)(t) = f(t)$, $(I_q^v f)(t) = \frac{1}{\Gamma_q(v)} \int_0^t (t - qs)^{(v-1)} f(s) d_qs$, $v > 0$, $t \in [0, 1]$, 其中 $L^1([0, 1])$ 是由 $[0, 1]$ 可测函数构成的经典 Banach 空间.

定义 2^[5] 令 $v \geq 0$, 定义 Riemann-Liouville 型 q -导数为 $D_q^0 f(t) = f(t)$, $(D_q^v f)(t) = (D_q^n I_q^{n-v} f)(t)$, 其中 n 是大于等于 v 的最小整数.

定义 3^[6] 令 $v \geq 0$, 并定义 Caputo 型 q -导数为 $({}^C D_q^v f)(t) = (I_q^{n-v} D_q^n f)(t)$.

引理 1^[6] 令 $v > 0$, n 是大于等于 v 的最小整数, 则 $(I_q^v {}^C D_q^v f)(t) = \sum_{k=0}^{n-1} \frac{t^k}{\Gamma_q(k+1)} D_q^k f(0)$.

引理 2^[6] 令 $\alpha \in \mathbf{R}^+$, $\beta \in (-1, +\infty)$, 则 $I_q^\alpha (t-s)^{(\beta)} = \frac{\Gamma_q(\beta+1)}{\Gamma_q(\beta+\alpha+1)} (t-s)^{(\beta+\alpha)}$.

引理 3^[7] 令 X 为 Banach 空间, $\Omega \subset X$ 是一个凸的有界闭集. 如果 $T: \Omega \rightarrow \Omega$ 是 $T\Omega \subset X$ 的连续算子, 且 $T\Omega$ 是相对紧的, 则 T 在 Ω 中至少有一个不动点.

根据方程(1), 令 $\phi_p({}^C D_q^\beta u(t)) = v(t)$, 则 $v(1) = b^{p-1} v(\xi)$. 下面考虑如下方程:

$$\begin{cases} -{}^C D_q^\beta v(t) = y(t), & t \in [0, 1]; \\ v(1) = b^{p-1} v(\xi). \end{cases} \quad (2)$$

引理 4 令 $y \in C[0, 1]$, 则方程(2) 有唯一解 $v(t) = \int_0^1 H(t, q\tau) y(\tau) d_q\tau$, 其中

$$H(t, q\tau) = \begin{cases} \frac{(1-q\tau)^{(\beta-1)}}{(1-b^{p-1})\Gamma_q(\beta)} - \frac{(t-q\tau)^{(\beta-1)}}{\Gamma_q(\beta)}, & 0 \leq q\tau \leq t \leq 1, \xi \leq q\tau; \\ \frac{(1-q\tau)^{(\beta-1)}}{(1-b^{p-1})\Gamma_q(\beta)} - \frac{(t-q\tau)^{(\beta-1)}}{\Gamma_q(\beta)} - \frac{b^{p-1}(\xi-q\tau)^{(\beta-1)}}{(1-b^{p-1})\Gamma_q(\beta)}, & 0 \leq q\tau \leq t \leq 1, \xi \geq q\tau; \\ \frac{(1-q\tau)^{(\beta-1)}}{(1-b^{p-1})\Gamma_q(\beta)}, & 0 \leq t \leq q\tau \leq 1, \xi \leq q\tau; \\ \frac{(1-q\tau)^{(\beta-1)}}{(1-b^{p-1})\Gamma_q(\beta)} - \frac{b^{p-1}(\xi-q\tau)^{(\beta-1)}}{(1-b^{p-1})\Gamma_q(\beta)}, & 0 \leq t \leq q\tau \leq 1, \xi \geq q\tau. \end{cases} \quad (3)$$

证明 假设 $v(t)$ 满足边值问题(2), 则由引理 1 可得 $v(t) = -\frac{1}{\Gamma_q(\beta)} \int_0^t (t-qs)^{(\beta-1)} y(s) d_qs - c_0$.

再利用边界条件 $(v(1) = b^{p-1} v(\xi))$ 可得:

$$c_0 = \frac{b^{p-1}}{(1-b^{p-1})\Gamma_q(\beta)} \int_0^\xi (\xi-qs)^{(\beta-1)} y(s) d_qs - \frac{1}{(1-b^{p-1})\Gamma_q(\beta)} \int_0^1 (1-qs)^{(\beta-1)} y(s) d_qs.$$

由上式可得:

$$\begin{aligned} v(t) &= -\frac{1}{\Gamma_q(\beta)} \int_0^t (t-qs)^{(\beta-1)} y(s) d_qs - \frac{b^{p-1}}{(1-b^{p-1})\Gamma_q(\beta)} \int_0^\xi (\xi-qs)^{(\beta-1)} y(s) d_qs + \\ &\quad \frac{1}{(1-b^{p-1})\Gamma_q(\beta)} \int_0^1 (1-qs)^{(\beta-1)} y(s) d_qs = \int_0^1 H(t, qs) y(s) d_qs. \end{aligned}$$

证毕.

由上述证明可知方程(4) 等价于方程(5).

$$\begin{cases} {}^C D_q^\beta (\phi_p({}^C D_q^\alpha u(t))) + y(t) = 0, & t \in [0, 1]; \\ u(1) = \lambda \int_0^1 u(s) d_qs, & D_q u(0) = 0; \\ {}^C D_q^\alpha u(1) = b {}^C D_q^\alpha u(\xi). \end{cases} \quad (4)$$

$$\begin{cases} {}^c D_q^\alpha u(t) = \phi_{q^*} \int_0^1 H(t, q\tau) y(\tau) d_q \tau, & 0 < t < 1; \\ D_q u(0) = 0, u(1) = \lambda \int_0^1 u(s) d_q s. \end{cases} \quad (5)$$

引理 5 令 $y \in C[0, 1]$, 则边值问题(5) 有如下唯一解:

$$\begin{aligned} u(t) = & \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha-1)} \phi_{q^*} \left(\int_0^1 H(t, q\tau) y(\tau) d_q \tau \right) d_q s - \\ & \frac{1}{\Gamma_q(\alpha)} \int_0^1 (1 - qs)^{(\alpha-1)} \phi_{q^*} \left(\int_0^1 H(s, q\tau) y(\tau) d_q \tau \right) d_q s + \lambda A, \end{aligned} \quad (6)$$

$$\text{其中 } A = \frac{1}{1-\lambda} \left[\frac{1}{[\alpha]_q \Gamma_q(\alpha)} \int_0^1 (1 - qs)^{(\alpha)} \phi_{q^*} \left(\int_0^1 H(s, q\tau) y(\tau) d_q \tau \right) d_q s - \frac{1}{\Gamma_q(\alpha)} \int_0^1 (1 - qs)^{(\alpha-1)} \phi_{q^*} \cdot \left(\int_0^1 H(s, q\tau) y(\tau) d_q \tau \right) d_q s \right].$$

证明 由引理 1 可知 $u(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha-1)} \phi_{q^*} \left(\int_0^1 H(s, q\tau) y(\tau) d_q \tau \right) d_q s - c_1 - c_2 t$, 于是有

$$\begin{aligned} D_q u(t) = & \frac{1}{(1-q)t \Gamma_q(\alpha)} \left[\int_0^t (t - qs)^{(\alpha-1)} \phi_{q^*} \left(\int_0^1 H(s, q\tau) y(\tau) d_q \tau \right) d_q s - \right. \\ & \left. \int_0^{qt} (qt - qs)^{(\alpha-1)} \phi_{q^*} \left(\int_0^1 H(s, q\tau) y(\tau) d_q \tau \right) d_q s \right] - c_2. \end{aligned}$$

利用边界条件 ($D_q u(0) = 0$) 可得 $c_2 = 0$, $u(1) = \lambda \int_0^1 u(s) d_q s$, $u(1) = \frac{1}{\Gamma_q(\alpha)} \int_0^1 (1 - qs)^{(\alpha-1)} \phi_{q^*} \cdot$

$\left(\int_0^1 H(t, q\tau) y(\tau) d_q \tau \right) d_q s - c_1$. 由此可得

$$c_1 = \frac{1}{\Gamma_q(\alpha)} \int_0^1 (1 - qs)^{(\alpha-1)} \phi_{q^*} \left(\int_0^1 H(s, q\tau) y(\tau) d_q \tau \right) d_q s - \lambda \int_0^1 u(s) d_q s.$$

令

$$\begin{aligned} A = & \int_0^1 u(t) d_q t = \int_0^1 \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \phi_{q^*} \left(\int_0^1 H(s, q\tau) y(\tau) d_q \tau \right) d_q s d_q t - \\ & \int_0^1 \int_0^1 \frac{(1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \phi_{q^*} \left(\int_0^1 H(s, q\tau) y(\tau) d_q \tau \right) d_q s d_q t + \int_0^1 \lambda A d_q t = \\ & \frac{1}{[\alpha]_q \Gamma_q(\alpha)} \int_0^1 (1 - qs)^{(\alpha)} \phi_{q^*} \left(\int_0^1 H(s, q\tau) y(\tau) d_q \tau \right) d_q s - \\ & \frac{1}{\Gamma_q(\alpha)} \int_0^1 (1 - qs)^{(\alpha-1)} \phi_{q^*} \left(\int_0^1 H(s, q\tau) y(\tau) d_q \tau \right) d_q s + \lambda A, \end{aligned}$$

则有 $A = \frac{1}{1-\lambda} \left[\frac{1}{[\alpha]_q \Gamma_q(\alpha)} \int_0^1 (1 - qs)^{(\alpha)} \phi_{q^*} \left(\int_0^1 H(s, q\tau) y(\tau) d_q \tau \right) d_q s - \frac{1}{\Gamma_q(\alpha)} \int_0^1 (1 - qs)^{(\alpha-1)} \phi_{q^*} \cdot \left(\int_0^1 H(s, q\tau) y(\tau) d_q \tau \right) d_q s \right]$, 进而可得式(6) 成立. 证毕.

引理 6 函数 H 为 $[0, 1] \times [0, 1]$ 上的连续函数, 且有以下性质:

① $H(t, q\tau) \leq H(q\tau, q\tau)$, $t, q\tau \in [0, 1]$;

② $\int_0^1 H(q\tau, q\tau) d_q \tau \leq \frac{1}{1-b^{p-1}} \Gamma(\beta+1)$.

证明 ① 对于任意的 $t, q\tau \in [0, 1]$, 由式(3) 可知 $H(t, q\tau) \leq H(q\tau, q\tau)$. ② 对于任意的 $t, q\tau \in [0, 1]$, 通过式(3) 和 q -积分的定义可得:

$$\int_0^1 H(q\tau, q\tau) d_q \tau \leq \int_0^1 \frac{(1 - q\tau)^{(\beta-1)}}{(1-b^{p-1}) \Gamma_q(\beta)} d_q \tau = \frac{1}{(1-b^{p-1}) \Gamma_q(\beta)} \int_0^1 (1 - q\tau)^{(\beta-1)} d_q \tau =$$

$$\frac{1}{(1-b^{p-1})\Gamma_q(\beta)} \cdot \frac{\Gamma_q(\beta)}{\Gamma_q(\beta+1)} (1-q\tau)^{(\beta)} \leq \frac{1}{(1-b^{p-1})\Gamma_q(\beta+1)}.$$

证毕.

2 主要结果及其证明

下面利用 Schauder 不动点定理证明边值问题(1) 的解的存在性. 首先令 $I = [0, 1]$, $U = \{u(t) \mid u(t) \in C(I)\}$, 并定义范数 $\|u\| = \max_{t \in [0, 1]} |u(t)|$, $(U, \|\cdot\|)$ 为 Banach 空间.

定理 1 假设条件 (H_1) 与 (H_2) 成立, 则问题(1) 至少存在一个解.

(H_1) $f(t, u): [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ 是连续函数.

(H_2) 存在常数 $k > 0$, 有 $f(t, u) \leq L\phi_p(u)$, $t \in [0, 1]$, $\|u\| \leq k$, 其中

$$0 < L \leq \phi_p \left(\frac{(1-\lambda)[\alpha]_q \Gamma_q(\alpha)}{((1-\lambda)[\alpha]_q + \lambda)(1-q) \sum_{n=0}^{\infty} (1-q)^{(\alpha)} q^n} \right) \left(\frac{1}{(1-b^{p-1})\Gamma_q(\beta+1)} \right)^{-1}.$$

证明 令 $P = \{u(t) \mid \|u(t)\| \leq k, t \in [0, 1]\}$, 则由此可知 $P \subset U$ 是凸的有界闭空间. 对于任意的 $u \in P$, 定义算子 $T: P \rightarrow U$, 并定义

$$\begin{aligned} Tu(t) = & \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} \phi_{q^*} \left(\int_0^1 H(t, q\tau) f(\tau, u(\tau)) d_q \tau \right) d_q s - \\ & \frac{1}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} \phi_{q^*} \left(\int_0^1 H(s, q\tau) f(\tau, u(\tau)) d_q \tau \right) d_q s + \\ & \frac{\lambda}{1-\lambda} \left[\frac{1}{[\alpha]_q \Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha)} \phi_{q^*} \left(\int_0^1 H(s, q\tau) f(\tau, u(\tau)) d_q \tau \right) d_q s - \right. \\ & \left. \frac{1}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} \phi_{q^*} \left(\int_0^1 H(s, q\tau) f(\tau, u(\tau)) d_q \tau \right) d_q s \right], \end{aligned}$$

于是由假设 (H_2) 可得 $f(t, u) \leq L\phi_p(u) \leq L\phi_p(k)$. 下面证明算子 T 的一致有界性.

$$\begin{aligned} |Tu(t)| \leq & \left| \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \phi_{q^*} \left(\int_0^1 H(q\tau, q\tau) f(\tau, u(\tau)) d_q \tau \right) d_q s - \right. \\ & \frac{1}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} \phi_{q^*} \left(\int_0^1 H(q\tau, q\tau) f(\tau, u(\tau)) d_q \tau \right) d_q s + \\ & \frac{\lambda}{1-\lambda} \left[\frac{1}{[\alpha]_q \Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha)} \phi_{q^*} \left(\int_0^1 H(q\tau, q\tau) f(\tau, u(\tau)) d_q \tau \right) d_q s - \right. \\ & \left. \frac{1}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} \phi_{q^*} \left(\int_0^1 H(q\tau, q\tau) f(\tau, u(\tau)) d_q \tau \right) d_q s \right] \Big| \leq \\ & \left| \frac{t(1-q)}{\Gamma_q(\alpha)} \sum_{n=0}^{\infty} (t-tq^{n+1})^{(\alpha-1)} q^n \phi_{q^*} \left(\int_0^1 H(q\tau, q\tau) f(\tau, u(\tau)) d_q \tau \right) - \right. \\ & \frac{1-q}{\Gamma_q(\alpha)} \sum_{n=0}^{\infty} (1-q^{n+1})^{(\alpha-1)} q^n \phi_{q^*} \left(\int_0^1 H(q\tau, q\tau) f(\tau, u(\tau)) d_q \tau \right) + \\ & \frac{\lambda}{1-\lambda} \left[\frac{1-q}{[\alpha]_q \Gamma_q(\alpha)} \sum_{n=0}^{\infty} (1-q^{n+1})^{(\alpha)} q^n \phi_{q^*} \left(\int_0^1 H(q\tau, q\tau) f(\tau, u(\tau)) d_q \tau \right) - \right. \\ & \left. \frac{1-q}{\Gamma_q(\alpha)} \sum_{n=0}^{\infty} (1-q^{n+1})^{(\alpha-1)} q^n \phi_{q^*} \left(\int_0^1 H(q\tau, q\tau) f(\tau, u(\tau)) d_q \tau \right) \right] \Big| \leq \\ & \left| \frac{t(1-q)}{\Gamma_q(\alpha)} \sum_{n=0}^{\infty} (t-tq^{n+1})^{(\alpha-1)} q^n \phi_{q^*} \left(\int_0^1 H(q\tau, q\tau) f(\tau, u(\tau)) d_q \tau \right) + \right. \\ & \left. \frac{\lambda(1-q)}{(1-\lambda)[\alpha]_q \Gamma_q(\alpha)} \sum_{n=0}^{\infty} (1-q^{n+1})^{(\alpha)} q^n \phi_{q^*} \left(\int_0^1 H(q\tau, q\tau) f(\tau, u(\tau)) d_q \tau \right) \right| \leq \end{aligned}$$

$$\begin{aligned}
& \left| \frac{((1-\lambda)[\alpha]_q + \lambda)(1-q)}{(1-\lambda)[\alpha]_q \Gamma_q(\alpha)} \sum_{n=0}^{\infty} (1-q^{n+1})^{(\alpha)} q^n \phi_{q^*} \left(\int_0^1 H(q\tau, q\tau) L\phi_p(u) d_q\tau \right) \right| \leq \\
& \left| \frac{((1-\lambda)[\alpha]_q + \lambda)(1-q)}{(1-\lambda)[\alpha]_q \Gamma_q(\alpha)} \sum_{n=0}^{\infty} (1-q^{n+1})^{(\alpha)} q^n \phi_{q^*} \left(\int_0^1 H(q\tau, q\tau) L\phi_p(k) d_q\tau \right) \right| \leq \\
& \frac{((1-\lambda)[\alpha]_q + \lambda)(1-q)}{(1-\lambda)[\alpha]_q \Gamma_q(\alpha)} \sum_{n=0}^{\infty} (1-q^{n+1})^{(\alpha)} q^n \left| \phi_{q^*} \left(\int_0^1 H(q\tau, q\tau) d_q\tau \right) \right| \phi_{q^*}(L)k \leq \\
& \frac{((1-\lambda)[\alpha]_q + \lambda)(1-q)}{(1-\lambda)[\alpha]_q \Gamma_q(\alpha)} \sum_{n=0}^{\infty} (1-q^{n+1})^{(\alpha)} q^n \left| \phi_{q^*} \left(\frac{1}{(1-b^{p-1})\Gamma_q(\beta+1)} \right) \right| \phi_{q^*}(L)k \leq k,
\end{aligned}$$

因此 $T(P) \subseteq P$. 再由假设 (H_2) 可得

$$\left| \phi_{q^*} \left(\int_0^1 H(s, q\tau) f(\tau, u(\tau)) d_q\tau \right) \right| \leq \left| \phi_{q^*} \left(\frac{L\phi_p(k)}{(1-b^{p-1})\Gamma(\beta+1)} \right) \right| \leq \left(\frac{L\phi_p(k)}{(1-b^{p-1})\Gamma(\beta+1)} \right)^{q^{-1}}.$$

令 $M := \left(\frac{L\phi_p(k)}{(1-b^{p-1})\Gamma(\beta+1)} \right)^{q^{-1}}$, 则对于任意的 $t_1, t_2 \in [0, 1]$, $t_1 < t_2$ 有:

$$\begin{aligned}
|Tu(t_2) - Tu(t_1)| & \leq \left| \int_0^{t_2} \frac{(t_2 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \phi_{q^*} \left(\int_0^1 H(q\tau, q\tau) f(\tau, u(\tau)) d_q\tau \right) d_qs - \right. \\
& \left. \int_0^{t_1} \frac{(t_1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \phi_{q^*} \left(\int_0^1 H(q\tau, q\tau) f(\tau, u(\tau)) d_q\tau \right) d_qs \right| \leq \left(\frac{L\phi_p(k)}{(1-b^{p-1})\Gamma(\beta+1)} \right)^{q^{-1}} \times \\
& \left| \frac{t_2(1-q) \sum_{n=0}^{\infty} (t_2 - t_2 q^{n+1})^{(\alpha-1)} q^n - t_1(1-q) \sum_{n=0}^{\infty} (t_1 - t_1 q^{n+1})^{(\alpha-1)} q^n}{\Gamma_q(\alpha)} \right| = \\
& \left(\frac{L\phi_p(k)}{(1-b^{p-1})\Gamma(\beta+1)} \right)^{q^{-1}} \left| \frac{t_2(1-q)}{\Gamma_q(\alpha)} \sum_{n=0}^{\infty} (t_2)^{\alpha-1} \prod_{i=0}^{\infty} \frac{t_2 - t_2 q^{n+1} q^i}{t_2 - t_2 q^{n+1} q^{i+\alpha-1}} q^n - \right. \\
& \left. \frac{t_1(1-q)}{\Gamma_q(\alpha)} \sum_{n=0}^{\infty} (t_1)^{\alpha-1} \prod_{i=0}^{\infty} \frac{t_1 - t_1 q^{n+1} q^i}{t_1 - t_1 q^{n+1} q^{i+\alpha-1}} q^n \right| = \\
& M \left| \frac{(t_2^{\alpha-1} + t_2^{\alpha-2} t_1 + t_2^{\alpha-3} t_1^2 + \cdots + t_2 t_1^{\alpha-2} + t_1^{\alpha-1})(1-q)}{\Gamma_q(\alpha)} \sum_{n=0}^{\infty} \prod_{i=0}^{\infty} \frac{1 - q^{n+1} q^i}{1 - q^{n+1} q^{i+\alpha-1}} q^n \right| |t_2 - t_1|.
\end{aligned}$$

由上式可知, 当 $t_2 \rightarrow t_1$ 时, 上述不等式右侧可在不依赖于函数 u 的条件下趋近于 0, 因此 $T(P)$ 是等度连续的. 根据 Arzelà-Ascoli 定理可知, T 是紧的. 再根据 Schauder 不动点定理可知, T 在 $u \in P$ 时至少存在一个不动点, 因此问题 (1) 在 P 中至少有一个正解. 证毕.

3 算例

例 1 考虑如下分数阶边值问题:

$$\begin{cases} {}^c D_{1/2}^{1/2}(\phi_{4/3}({}^c D_{1/2}^{3/2} u(t))) = \frac{\phi_{4/3}|u(t)|}{(t+5)^2}, & t \in (0, 1); \\ u(1) = \frac{1}{2} \int_0^1 u(s) d_{1/2}s, & D_{1/2} u(0) = 0; \\ {}^c D_{1/2}^{3/2} u(1) = \frac{1}{2} {}^c D_q^{3/2} u\left(\frac{1}{2}\right). \end{cases} \quad (7)$$

其中 $\beta = \frac{1}{2}$, $\alpha = \frac{3}{2}$, $p = \frac{4}{3}$, $q^* = 4$, $q = \lambda = \xi = b = \frac{1}{2}$, $f(t, u) = \frac{\phi_{4/3}|u(t)|}{(t+5)^2}$. 因为 f 是连续的, 且

$|f(t, u)| \leq \frac{1}{25} \phi_p |u(t)|$, $(t, u) \in [0, 1] \times [0, \infty)$, 所以由条件 (H_2) 可得 $L = \frac{1}{25}$, 并进而可得

$$\phi_p \left(\frac{(1-\lambda)[\alpha]_q \Gamma_q(\alpha)}{((1-\lambda)[\alpha]_q + \lambda)(1-q) \sum_{n=0}^{\infty} (1-q)^{(a)} q^n} \right) \left(\frac{1}{(1-b^{p-1}) \Gamma_q(\beta+1)} \right)^{-1} \approx 0.4614.$$

由以上显然可得:

$$0 < L \leq \phi_p \left(\frac{(1-\lambda)[\alpha]_q \Gamma_q(\alpha)}{((1-\lambda)[\alpha]_q + \lambda)(1-q) \sum_{n=0}^{\infty} (1-q)^{(a)} q^n} \right) \left(\frac{1}{(1-b^{p-1}) \Gamma_q(\beta+1)} \right)^{-1}.$$

再由定理 1 可知,边值问题(7)至少存在一个解.

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