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一类 Hadamard 型分数阶微分方程解的存在唯一性

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摘要: 利用格林函数的性质和锥上不动点定理讨论了一类 Hadamard 型分数阶微分方程(非线性项包含分数阶导数和一个减算子)的正解，得到了该分数阶微分方程正解的存在唯一性.

关键词: Hadamard 型分数阶微分方程; 格林函数; 混合单调算子; 不动点定理; 存在唯一性
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The existence and uniqueness of the solutions of a class of Hadamard type fractional differential equations

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Abstract: Using the properties of Green's function and the fixed point theorem on the cone, the positive solutions of a class of Hadamard type fractional differential equations (the nonlinear term includes fractional derivatives and a subtraction operator) are discussed, and the existence and uniqueness of positive solutions of the fractional differential equations is obtained.

Keywords: Hadamard type fractional differential equation; Green's function; mixed monotone operator; fixed point theorem; existence and uniqueness

0 引言

分数阶微积分方程在许多自然科学领域有着广泛的应用. 近年来，一些学者研究了带有边值问题微分方程正解的存在唯一性，并取得了较好的研究成果[1-5]. 本文研究如下一类 Hadamard 型具有导数项的分数阶微分方程正解的存在唯一性:

$$\begin{cases} {}^{\mathrm{H}}\mathrm{D}_{1+}^{\alpha}u(t)+p(t)f(t,u(t),{}^{\mathrm{H}}\mathrm{D}_{1+}^{\beta}u(t))+q(t)g(t,u(t),(Nu)(t))=0; \\ u^{(i)}(1)=0,\ i=0,\cdots,n-2; \\ [{}^{\mathrm{H}}\mathrm{D}_{1+}^{\gamma}u(t)]_{t=e}=k(u(e)). \end{cases} \tag{1}$$

其中: $t\in(1,e)$; $n-1<\alpha<n\ (n\in\mathbf{N},n>3)$; $1<\beta\leqslant\gamma\leqslant n-2$; $p,q\in C([1,e],[0,\infty))$; ${}^{\mathrm{H}}\mathrm{D}_{1+}^{\alpha}$ 和 ${}^{\mathrm{H}}\mathrm{D}_{1+}^{\beta}$ 为 Hadamard 型导数; $f:[1,e]\times\mathbf{R}_{+}\times\mathbf{R}_{+}\to\mathbf{R}_{+}$, $g:[1,e]\times\mathbf{R}_{+}\times\mathbf{R}_{+}\to\mathbf{R}_{+}$ 和 $k:[0,\infty)\to[0,\infty)$ 均为连续函数; $N:\mathbf{R}_{+}\to\mathbf{R}_{+}$ 为减算子; $p(t),q(t):\mathbf{R}_{+}\to\mathbf{R}_{+}$ 为连续函数.

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1 相关知识和引理

定义 1[1] 函数 $g:[1,+\infty)\to \mathbf{R}_+$, $g\in L^1[1,+\infty)$ 的 Hadamard 型 $\alpha\in\mathbf{R}_+$ 阶分数阶积分为

$$ {}^{\mathrm{H}}\mathrm{I}_{1+}^{\alpha}g(t)=\frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}g(s)\frac{1}{s}\mathrm{d}s. \tag{2}$$

定义 2[1] 函数 $g:[1,+\infty)\to\mathbf{R}_+$ 的 $\alpha\in\mathbf{R}_+$ 阶的 Hadamard 型分数阶导数为

$$ {}^{\mathrm{H}}\mathrm{D}_{1+}^{\alpha}g(t)=\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)^n\frac{1}{\Gamma(n-\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{n-\alpha-1}g(s)\frac{1}{s}\mathrm{d}s,\ n-1<\alpha<n,\ n=[\alpha]+1. \tag{3}$$

引理 1[1] 若 $\beta-1>\gamma\geqslant 0$, $t>a>1$, 则有

$$ {}^{\mathrm{H}}\mathrm{D}_{1+}^{\gamma}\left(\ln\frac{t}{a}\right)^{\beta-1}=\frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)}\left(\ln\frac{t}{a}\right)^{\beta-\gamma-1}. \tag{4}$$

引理 2[1] 令 $x\in C[1,\infty)\cap[1,\infty)$, 则有 ${}^{\mathrm{H}}\mathrm{I}_{1+}^{\gamma}({}^{\mathrm{H}}\mathrm{D}_{1+}^{\gamma}x(t))=x(t)-\sum_{i=1}^{n}c_i(\ln t)^{\gamma-i}$, 其中 $n=[\gamma]+1$, $c_i\in\mathbf{R}$.

引理 3 设 $h(t)\in C[1,\infty)$, 则边值问题

$$\begin{cases} {}^{\mathrm{H}}\mathrm{D}_{1+}^{\alpha}u(t)+h(t)=0,\ t\in(1,e),\ n-1<\alpha<n; \\ u^{(i)}(1)=0,\ i=0,\cdots,n-2; \\ {}^{\mathrm{H}}\mathrm{D}_{1+}^{\gamma}u(t)\,|_{t=e}=k(u(e)),\ 1\leqslant\gamma\leqslant n-2 \end{cases} \tag{5}$$

有唯一的解 $u(t)=\int_1^e G(t,s)h(s)\frac{\mathrm{d}s}{s}+\frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}k(u(e))(\ln t)^{\alpha-1}$, 其中

$$G(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}(\ln t)^{\alpha-1}(1-\ln s)^{\alpha-\gamma-1}-(\ln t-\ln s)^{\alpha-1},\ 1\leqslant s\leqslant t\leqslant e; \\ (\ln t)^{\alpha-1}(1-\ln s)^{\alpha-\gamma-1},\ 1\leqslant t\leqslant s\leqslant e.\end{cases} \tag{6}$$

证明 由引理 2 可得 $u(t)=-{}^{\mathrm{H}}\mathrm{I}_{1+}^{\alpha}h(t)+\sum_{i=1}^{n}c_i(\ln t)^{\alpha-i}$, 其中 $n=[\alpha]+1$, $c_i\in\mathbf{R}$. 再根据边值条件 $u^{(i)}(1)=0$ 可推出 $c_j=0$, $j=2,\cdots,n-2$, 因此有

$$u(t)=-{}^{\mathrm{H}}\mathrm{I}_{1+}^{\alpha}h(t)+c_1(\ln t)^{\alpha-1}. \tag{7}$$

由 ${}^{\mathrm{H}}\mathrm{D}_{1+}^{\gamma}u(t)\,|_{t=e}=k(u(e))$, 可得 $-{}^{\mathrm{H}}\mathrm{I}_{1+}^{\alpha-\gamma}h(e)+c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma)}=k(u(e))$, 故 $c_1=\frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}(k(u(e))+{}^{\mathrm{H}}\mathrm{I}_{1+}^{\alpha-\gamma}h(e))$. 将 c_1 代入式(7), 得

$$u(t)=-{}^{\mathrm{H}}\mathrm{I}_{1+}^{\alpha}h(t)+\frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}(k(u(e))+{}^{\mathrm{H}}\mathrm{I}_{1+}^{\alpha-\gamma}h(e))(\ln t)^{\alpha-1}=$$
$$\int_1^e G(t,s)h(s)\frac{\mathrm{d}s}{s}+\frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}k(u(e))(\ln t)^{\alpha-1}.$$

引理 4[6] 由式(6)定义的格林函数具有如下性质:

1) $G(t,s):[1,e]\times[1,e]\to[0,\infty)$ 是连续的;

2) $\forall t,s\in[1,e]$, 有:

$$0\leqslant(\ln t)^{\alpha-1}(1-\ln s)^{\alpha-\gamma-1}[1-(1-\ln s)^{\gamma}]\leqslant\Gamma(\alpha)G(t,s)\leqslant(\ln t)^{\alpha-1}(1-\ln s)^{\alpha-\gamma-1}, \tag{8}$$

$$0\leqslant(\ln t)^{\alpha-\beta-1}(1-\ln s)^{\alpha-\gamma-1}[1-(1-\ln s)^{\gamma}]\leqslant {}^{\mathrm{H}}\mathrm{D}_{1+}^{\beta}\Gamma(\alpha-\beta)G(t,s)\leqslant (\ln t)^{\alpha-\beta-1}(1-\ln s)^{\alpha-\gamma-1}. \tag{9}$$

引理 5[1] 设 P 是一个正规锥, $A,B:P\times P\to P$ 是两个混合单调算子, 且 $C:P\to P$ 是一个减算子. 假设以下条件成立:

(A1) $\forall t\in(0,1)$, 存在 $\varphi(t)\in(0,1]$, 使得 $A(tx,t^{-1}y)\geqslant\varphi(t)A(x,y)$, $\forall(x,y)\in P$;

(A2) $\forall t\in(0,1)$,有 $B(tx,t^{-1}y)\geqslant tB(x,y)$,$\forall x,y\in P$;

(A3) $\forall t\in(0,1)$,有 $C(t^{-1}y)\geqslant tC(y)$,$\forall y\in P$;

(A4) $\exists h>\theta$ 且 $h\in P_h$,使得 $A(h,h)\in P_h$,$B(h,h)\in P_h$,$C(h)\in P_h$;

(A5) $\exists\delta>0$,使得 $\forall x,y\in P$,有 $A(x,y)\geqslant\delta(B(x,y)+C(y))$.

根据以上假设有:

(1)$A:P_h\times P_h\to P_h$, $B:P_h\times P_h\to P_h$, $C:P_h\to P_h$;

(2)$\exists u_0,v_0\in P_h$,且存在 $r\in(0,1)$,使得 $rv_0\leqslant u_0\leqslant v_0$, $u_0=A(u_0,v_0)+B(u_0,v_0)+C(v_0)\leqslant A(v_0,u_0)+B(v_0,u_0)+C(u_0)\leqslant v_0$;

(3)$A(x,x)+B(x,x)+C(x)=x$ 有唯一的解 $x^*\in P_h$;

(4)$\forall x_0,y_0\in P_h$,依次构造如下序列:

$$x_n=A(x_{n-1},y_{n-1})+B(x_{n-1},y_{n-1})+C(y_{n-1}),\ n=1,2,\cdots;$$

$$y_n=A(y_{n-1},x_{n-1})+B(y_{n-1},x_{n-1})+C(x_{n-1}),\ n=1,2,\cdots,$$

则有 $x_n\to x_*$, $y_n\to y_*(n\to\infty)$.

2　主要结果及其证明

设 $E=C[1,e]$, $\|x(t)\|=\max\limits_{t\in[1,e]}|x(t)|$,显然$(E,\|\cdot\|)$是 Banach空间. 令 $P=\{x\in E\mid x(t)\geqslant0,{}^{\mathrm{H}}\mathrm{D}_{1+}^{\beta}x(t)\geqslant0,\forall t\in(1,e)\}$,由此易知 $P\subset E$,且 P 为正规锥. 给出如下 P 的半序:如果 $x(t)\leqslant y(t)$, ${}^{\mathrm{H}}\mathrm{D}_{1+}^{\beta}x(t)\leqslant{}^{\mathrm{H}}\mathrm{D}_{1+}^{\beta}y(t)$,则有 $x(t)\leq y(t)$. 由引理 3 可知,问题(1) 有唯一的解:

$$u(t)=\int_1^e G(t,s)p(s)f(s,u(s),{}^{\mathrm{H}}\mathrm{D}_{1+}^{\beta}u(s))\frac{\mathrm{d}s}{s}+$$
$$\int_1^e G(t,s)q(s)g(s,u(s),Nu(s))\frac{\mathrm{d}s}{s}+\frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}k(u(e))(\ln t)^{\alpha-1},\tag{10}$$

其中 $G(t,s)$ 与式(6) 相同.

定理 1　假设以下条件成立:

(H1) $f(t,x,y),g(t,x,y):[1,e]\times\mathbf{R}_+\times\mathbf{R}_+\to\mathbf{R}_+$ 均为连续函数,且关于第 2 个变量都是单调递增的,关于第 3 个变量都是单调递减的. $k(y):\mathbf{R}_+\to\mathbf{R}_+$ 为连续的单调递减函数, $N:\mathbf{R}_+\to\mathbf{R}_+$ 为减算子.

(H2) 当 $t\in(1,e)$ 时, $f\left(t,0,\dfrac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}\right)\neq0$, $g(t,0,N(0))\neq0$;当 $y\in\mathbf{R}_+$ 时, $k(y(e))\neq0$.

(H3) $\exists\gamma\in(0,1)$,使得 $\forall t\in(1,e)$, $\lambda\in(0,1)$,有 $f(t,\lambda x,\lambda^{-1}y)\geqslant\lambda^{\gamma}f(t,x,y)$, $g(t,\lambda x,\lambda^{-1}y)\geqslant\lambda g(t,x,y)$, $k(\lambda^{-1}y)\geqslant\lambda k(y)$, $N(\lambda u)\geqslant\lambda N(u)$.

(H4) $\forall t\in(1,e)$, $x\in\mathbf{R}_+$, $y\in\mathbf{R}_+$,存在常数 $\delta_1,\delta_2>0$,有 $f(t,x,y)\geqslant\delta_1 g(t,x,0)$, $f(t,x,y)\geqslant\delta_2 k(y)$.

(H5) $p(t),q(t):(1,e)\to[0,\infty)$ 为连续函数,且 $p(t)\geqslant q(t)\geqslant m>0$,其中 $m\in\mathbf{R}_+$.

则问题(1) 有唯一的解 $u^*\in P_h$, $h(t)=(\ln t)^{\alpha-1}$. 设初始值 $u_0,v_0\in P_h$,构造两组迭代序列,且 $u_n\to u_*$, $v_n\to v_*(n\to\infty)$,其中:

$$u_n(t)=\int_1^e G(t,s)p(s)f(s,u_{n-1}(s),{}^{\mathrm{H}}\mathrm{D}_{1+}^{\beta}v_{n-1}(s))\frac{\mathrm{d}s}{s}+$$
$$\int_1^e G(t,s)q(s)g(s,u_{n-1}(s),(Nv_{n-1})(s))\frac{\mathrm{d}s}{s}+\frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}k(v_{n-1}(e))(\ln t)^{\alpha-1},\ n=1,2,\cdots;$$

$$v_n(t)=\int_1^e G(t,s)p(s)f(s,v_{n-1}(s),{}^{\mathrm{H}}\mathrm{D}_{1+}^{\beta}u_{n-1}(s))\frac{\mathrm{d}s}{s}+$$

$$\int_1^e G(t,s)q(s)g(s,v_{n-1}(s),(Nu_{n-1})(s))\frac{\mathrm{d}s}{s}+\frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}k(u_{n-1}(e))(\ln t)^{\alpha-1},\ n=1,2,\cdots.$$

证明 定义以下算子:

$$A(u,v)(t)=\int_1^e G(t,s)p(s)f(s,u(s),{}^{\mathrm{H}}\mathrm{D}_{1+}^{\beta}v(s))\frac{\mathrm{d}s}{s},\tag{11}$$

$$B(u,v)(t)=\int_1^e G(t,s)q(s)g(s,u(s),Nv(s))\frac{\mathrm{d}s}{s},\tag{12}$$

$$C(v)(t)=\frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}k(v(e))(\ln t)^{\alpha-1},\tag{13}$$

$$T(u,v)=A(u,v)+B(u,v)+Cv.\tag{14}$$

为了方便证明,本文令 $u(t)\triangleq u$, $v(t)\triangleq v$, $(u,v)(t)\triangleq(u,v)$. 由式(9)可得到以下式子:

$${}^{\mathrm{H}}\mathrm{D}_{1+}^{\beta}A(u,v)=\int_1^e {}^{\mathrm{H}}\mathrm{D}_{1+}^{\beta}G(t,s)p(s)f(s,u,{}^{\mathrm{H}}\mathrm{D}_{1+}^{\beta}v)\frac{\mathrm{d}s}{s},\tag{15}$$

$${}^{\mathrm{H}}\mathrm{D}_{1+}^{\beta}B(u,v)=\int_1^e {}^{\mathrm{H}}\mathrm{D}_{1+}^{\beta}G(t,s)q(s)g(s,u,Nv)\frac{\mathrm{d}s}{s},\tag{16}$$

$${}^{\mathrm{H}}\mathrm{D}_{1+}^{\beta}C(v)=\frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha-\beta)}k(v(e))(\ln t)^{\alpha-\beta-1}.\tag{17}$$

由引理4和条件(H1)可知,$A(u,v)$, ${}^{\mathrm{H}}\mathrm{D}_{1+}^{\beta}A(u,v)>0$,因此 $A:P\times P\to P$. 同理可证,$B:P\times P\to P$, $C:P\to P$.

首先证明 A 和 B 是混合单调算子,C 是减算子. 实际上,对于 $u_1\geq u_2$, $v_1\geq v_2$,有 $u_1\geqslant u_2$, $v_1\geqslant v_2$, ${}^{\mathrm{H}}\mathrm{D}_{1+}^{\beta}u_1\geqslant{}^{\mathrm{H}}\mathrm{D}_{1+}^{\beta}u_2$, ${}^{\mathrm{H}}\mathrm{D}_{1+}^{\beta}v_1\geqslant{}^{\mathrm{H}}\mathrm{D}_{1+}^{\beta}v_2$,由此根据条件(H1)以及引理4易得:

$$A(u_1,v_2)=\int_1^e G(t,s)p(s)f(s,u_1,{}^{\mathrm{H}}\mathrm{D}_{1+}^{\beta}v_2)\frac{\mathrm{d}s}{s}\geqslant\int_1^e G(t,s)p(s)f(s,u_2,{}^{\mathrm{H}}\mathrm{D}_{1+}^{\beta}v_2)\frac{\mathrm{d}s}{s}\geqslant$$

$$\int_1^e G(t,s)p(s)f(s,u_2,{}^{\mathrm{H}}\mathrm{D}_{1+}^{\beta}v_1)\frac{\mathrm{d}s}{s}=A(u_2,v_1),$$

$${}^{\mathrm{H}}\mathrm{D}_{1+}^{\beta}A(u_1,v_2)=\int_1^e {}^{\mathrm{H}}\mathrm{D}_{1+}^{\beta}G(t,s)p(s)f(s,u_1,{}^{\mathrm{H}}\mathrm{D}_{1+}^{\beta}v_2)\frac{\mathrm{d}s}{s}\geqslant$$

$$\int_1^e {}^{\mathrm{H}}\mathrm{D}_{1+}^{\beta}G(t,s)p(s)f(s,u_2,{}^{\mathrm{H}}\mathrm{D}_{1+}^{\beta}v_2)\frac{\mathrm{d}s}{s}\geqslant\int_1^e {}^{\mathrm{H}}\mathrm{D}_{1+}^{\beta}G(t,s)p(s)f(s,u_2,{}^{\mathrm{H}}\mathrm{D}_{1+}^{\beta}v_1)\frac{\mathrm{d}s}{s}=$$

$${}^{\mathrm{H}}\mathrm{D}_{1+}^{\beta}A(u_2,v_1).$$

即 $A(u_1,v_2)\geq A(u_2,v_1)$,故算子 A 是混合单调算子. 同理可证 B 是混合单调算子,C 是减算子.

由(H3)有 $A(\lambda u,\lambda^{-1}v)\geqslant\lambda^{\gamma}A(u,v)$, $B(\lambda u,\lambda^{-1}v)\geqslant\lambda B(u,v)$, $C(\lambda^{-1}v)\geqslant\lambda C(v)$,因此算子 A、B、C 满足引理5的条件(A1)—(A3).

令 $h=(\ln t)^{\alpha-1}$, $t\in(1,e)$,则 $\forall u,v\in P_h$ 存在 $\mu\geqslant1$,使得 $\mu^{-1}h\leqslant x,y\leqslant\mu h$. 结合引理3和 f 的单调性可证得:

$$A(h,h)\geqslant\frac{(\ln t)^{\alpha-1}}{\Gamma(\alpha)}\int_1^e(1-\ln s)^{\alpha-\gamma-1}[1-(1-\ln s)^{\gamma}]p(s)f(s,h,{}^{\mathrm{H}}\mathrm{D}_{1+}^{\beta}h)\frac{\mathrm{d}s}{s}\geqslant$$

$$\frac{(\ln t)^{\alpha-1}}{\Gamma(\alpha)}\int_1^e(1-\ln s)^{\alpha-\gamma-1}[1-(1-\ln s)^{\gamma}]p(s)f\left(s,0,\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}\right)\frac{\mathrm{d}s}{s},$$

$$A(h,h)\leqslant\frac{(\ln t)^{\alpha-1}}{\Gamma(\alpha)}\int_1^e(1-\ln s)^{\alpha-\gamma-1}p(s)f(s,h,{}^{\mathrm{H}}\mathrm{D}_{1+}^{\beta}h)\frac{\mathrm{d}s}{s}\leqslant$$

$$\frac{(\ln t)^{\alpha-1}}{\Gamma(\alpha)}\int_1^e(1-\ln s)^{\alpha-\gamma-1}p(s)f(s,1,0)\frac{\mathrm{d}s}{s},$$

$$ {}^{H}D_{1+}^{\beta}A(h,h) \geqslant \frac{{}^{H}D_{1+}^{\beta}(\ln t)^{\alpha-1}}{\Gamma(\alpha)}\int_{1}^{e}(1-\ln s)^{\alpha-\gamma-1}\left[1-(1-\ln s)^{\gamma}\right]p(s)f\left(s,0,\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}\right)\frac{ds}{s}, $$

$$ {}^{H}D_{1+}^{\beta}A(h,h) \leqslant \frac{{}^{H}D_{1+}^{\beta}(\ln t)^{\alpha-1}}{\Gamma(\alpha)}\int_{1}^{e}(1-\ln s)^{\alpha-\gamma-1}p(s)f(s,1,0)\frac{ds}{s}. $$

令：

$$ a_1=\frac{1}{\Gamma(\alpha)}\int_{1}^{e}(1-\ln s)^{\alpha-\gamma-1}\left[1-(1-\ln s)^{\gamma}\right]p(s)f\left(s,0,\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}\right)\frac{ds}{s},\ t\in(1,e); $$

$$ a_2=\frac{1}{\Gamma(\alpha)}\int_{1}^{e}(1-\ln s)^{\alpha-\gamma-1}p(s)f(s,1,0)\frac{ds}{s},\ t\in(1,e). $$

因为 $0<a_1\leqslant a_2<\infty$，因此可得 $a_1h\le A(h,h)\le a_2h$. 因 $h(t)\in(0,1)$，$t\in(1,e)$，则根据条件(H1)可得 $N(1)\leqslant N(h)\leqslant N(0)$. 由函数 g 的单调性可得：

$$ B(h,h)\geqslant\frac{(\ln t)^{\alpha-1}}{\Gamma(\alpha)}\int_{1}^{e}(1-\ln s)^{\alpha-\gamma-1}\left[1-(1-\ln s)^{\gamma}\right]q(s)g(s,h,Nh)\frac{ds}{s}\geqslant $$
$$ \frac{(\ln t)^{\alpha-1}}{\Gamma(\alpha)}\int_{1}^{e}(1-\ln s)^{\alpha-\gamma-1}\left[1-(1-\ln s)^{\gamma}\right]q(s)g(s,1,N(0))\frac{ds}{s}, $$

$$ B(h,h)\leqslant\frac{(\ln t)^{\alpha-1}}{\Gamma(\alpha)}\int_{1}^{e}(1-\ln s)^{\alpha-\gamma-1}q(s)g(s,h,Nh)\frac{ds}{s}\leqslant $$
$$ \frac{(\ln t)^{\alpha-1}}{\Gamma(\alpha)}\int_{1}^{e}(1-\ln s)^{\alpha-\gamma-1}q(s)g(s,h,N(1))\frac{ds}{s}, $$

$$ {}^{H}D_{1+}^{\beta}B(h,h)\geqslant\frac{{}^{H}D_{1+}^{\beta}(\ln t)^{\alpha-1}}{\Gamma(\alpha)}\int_{1}^{e}(1-\ln s)^{\alpha-\gamma-1}\left[1-(1-\ln s)^{\gamma}\right]q(s)g(s,1,N(0))\frac{ds}{s}, $$

$$ {}^{H}D_{1+}^{\beta}B(h,h)\leqslant\frac{{}^{H}D_{1+}^{\beta}(\ln t)^{\alpha-1}}{\Gamma(\alpha)}\int_{1}^{e}(1-\ln s)^{\alpha-\gamma-1}q(s)g(s,h,N(1))\frac{ds}{s}. $$

令：

$$ b_1=\frac{1}{\Gamma(\alpha)}\int_{1}^{e}(1-\ln s)^{\alpha-\gamma-1}\left[1-(1-\ln s)^{\gamma}\right]q(s)g(s,0,N(0))\frac{ds}{s},\ t\in(1,e); $$

$$ b_2=\frac{1}{\Gamma(\alpha)}\int_{1}^{e}(1-\ln s)^{\alpha-\gamma-1}q(s)\,g(s,1,N(1))\frac{ds}{s},\ t\in(1,e). $$

因为 $0<b_1\leqslant b_2<\infty$，因此可得 $b_1h\le B(h,h)\le b_2h$. 令 $c=k(v(e))\dfrac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}$，$\mu\geqslant\max\{1,a_1^{-1},a_2,b_1^{-1},b_2,c^{-1},c\}$，于是有 $\mu^{-1}h\leqslant A(h,h),B(h,h),C(h)\leqslant\mu h$，即 $A(h,h)\in P_h$，$B(h,h)\in P_h$，$C(h)\in P_h$. 该式满足引理 5 中的条件(A4).

最后证明算子 A、B、C 满足引理 5 中的条件(A5). 根据条件(H4)可得：

$$ A(u,v)=\int_{1}^{e}G(t,s)p(s)f(s,u,{}^{H}D_{1+}^{\beta}v)\frac{ds}{s}\geqslant\int_{1}^{e}G(t,s)q(s)\delta_1g(s,u,0)\frac{ds}{s}\geqslant $$
$$ \int_{1}^{e}G(t,s)q(s)\delta_1g(s,u,Nv)\frac{ds}{s}=\delta_1B(u,v), $$

$$ {}^{H}D_{1+}^{\beta}A(u,v)=\int_{1}^{e}{}^{H}D_{1+}^{\beta}G(t,s)p(s)f(s,u,{}^{H}D_{1+}^{\beta}v)\frac{ds}{s}\geqslant\int_{1}^{e}{}^{H}D_{1+}^{\beta}G(t,s)\delta_1q(s)g(s,u,0)\frac{ds}{s}\geqslant $$
$$ \int_{1}^{e}{}^{H}D_{1+}^{\beta}G(t,s)q(s)\delta_1g(s,u,Nv)\frac{ds}{s}=\delta_1{}^{H}D_{1+}^{\beta}B(u,v). $$

因此有 $A(u,v)\ge\delta_1B(u,v)$. 再根据条件(H3)和(H4)以及引理 4 可得：

$$ A(u,v)\geqslant\int_{1}^{e}G(t,s)mf(s,u,{}^{H}D_{1+}^{\beta}v)\frac{ds}{s}\geqslant $$

$$\frac{(\ln t)^{\alpha-1}}{\Gamma(\alpha)}\int_1^e (1-\ln s)^{\alpha-\gamma-1}[1-(1-\ln s)^{\gamma}]\,m\delta_2 k(v)\frac{\mathrm{d}s}{s}\geqslant$$

$$\frac{(\ln t)^{\alpha-1}}{\Gamma(\alpha)}\left(\frac{1}{\alpha-\gamma}-\frac{1}{\alpha}\right)m\delta_2 k(v(e))=m\delta_2\frac{1}{\Gamma(\alpha-\gamma)}\left(\frac{1}{\alpha-\gamma}-\frac{1}{\alpha}\right)C(v),$$

$${}^{\mathrm{H}}\mathrm{D}_{1+}^{\beta}A(u,v)\geqslant m\delta_2\frac{1}{\Gamma(\alpha-\gamma)}\left(\frac{1}{\alpha-\gamma}-\frac{1}{\alpha}\right){}^{\mathrm{H}}\mathrm{D}_{1+}^{\beta}C(v).$$

令 $2\delta=\min\left\{\delta_1,m\delta_2\frac{1}{\Gamma(\alpha-\gamma)}\left(\frac{1}{\alpha-\gamma}-\frac{1}{\alpha}\right)\right\}$,进而可得 $A(u,v)\geqslant\delta(B(u,v)+C(v))$.

综上,由引理 5 可知,算子 T 存在一个不动点 $u\in P_h$,满足 $T(u,u)=A(u,u)+B(u,u)+Cu=u$,因此问题(1)有唯一的解 $u\in P_h$, $h=(\ln t)^{\alpha-1}$, $t\in(1,e)$. 设初始值 $u_0,v_0\in P_h$,构造两组迭代序列 $\{u_n\}$ 和 $\{v_n\}$, $t\in(1,e)$:

$$u_n(t)=\int_1^e G(t,s)p(s)f(s,u_{n-1},{}^{\mathrm{H}}\mathrm{D}_{1+}^{\beta}v_{n-1})\frac{\mathrm{d}s}{s}+\int_1^e G(t,s)q(s)g(s,u_{n-1},Nv_{n-1})\frac{\mathrm{d}s}{s}+$$

$$\frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}k(v_{n-1})(\ln t)^{\alpha-1},\ n=1,2,\cdots;$$

$$v_n(t)=\int_1^e G(t,s)p(s)f(s,v_{n-1},{}^{\mathrm{H}}\mathrm{D}_{1+}^{\beta}u_{n-1})\frac{\mathrm{d}s}{s}+\int_1^e G(t,s)q(s)g(s,v_{n-1},Nu_{n-1})\frac{\mathrm{d}s}{s}+$$

$$\frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}k(u_{n-1}(e))(\ln t)^{\alpha-1},\ n=1,2,\cdots.$$

根据引理 5 可知,存在(u^*,v^*),满足 $u_n\to u^*$, $v_n\to v^*(n\to\infty)$,且 $u^*,v^*\in P_h$.

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