文章编号: 1004-4353(2020)02-0101-06

基于临界 Galton-Watson 过程的随机游动的大偏差

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摘要: 针对一族独立同分布的随机变量 $\{X_k\}$ 的和 $S_{Z_n}=\sum_{k=1}^{Z_n}X_k$($Z_n$ 为临界 Galton-Watson 过程的第 n 代个体数)，利用随机游动和概率论的知识研究了 $R_n:=S_{Z_n}/Z_n$ 的渐近性质以及在 $\{Z_n>0\}$ 条件下的 S_{Z_n} 的大偏差. 研究结果表明，R_n 的规范偏差概率有非退化的极限，并且其大偏差规范化后收敛到正常数.

关键词: 大偏差概率；临界 Galton-Watson 过程；条件概率；Fuk-Nagaev 不等式

中图分类号: O211.65　　　　**文献标识码:** A

Large deviation of random walk indexed by critical Galton-Watson process

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Abstract: Consider the sum $S_{Z_n}=\sum_{k=1}^{Z_n}X_k$ of a family of independent and identically distributed random variables $\{X_k\}$, where Z_n is the number of individuals of a critical Galton-Watson process in the generation n. The asymptotic properties of the ratio $R_n:=S_{Z_n}/Z_n$ and the large deviation of S_{Z_n} under $\{Z_n>0\}$ are studied by using the properties of random walks and branching processes. The proofs are given in detail that the normal deviation probability of R_n has nondegenerate limit and the large deviation probabilities of R_n after normalization converge to a positive constant.

Keywords: large deviation probability; critical Galton-Watson process; conditional probability; Fuk-Nagaev inequality

0　引言

对后代均值 m 的估计是研究 Galton-Watson 过程的一个重要内容. 令 $Z=(Z_n)_{n\geqslant 0}$ 是后代分布为 $\{p_k;k\geqslant 0\}$ 的 Galton-Watson 过程，且假设 Z 是临界的，即 $m:=\sum_{k=1}^{\infty}kp_k=1$. 在文献[1]中，Nagaev 提出了著名的 Lotka-Nogaev 估计，即用 $\frac{Z_{n+1}}{Z_n}$ 来估计 m. 如果 $\xi:=(\operatorname{Var}Z_1)^{1/2}\in(0,\infty)$，那么对所有 $x\in\mathbf{R}$，有

$$\lim_{n\uparrow\infty}\mathrm{P}\left(m^{n/2}\left(\frac{Z_{n+1}}{Z_n}-m\right)<x;Z_n>0\right)=\int_0^{\infty}\Phi\left(\frac{xu^{1/2}}{\xi}\right)\omega(u)\mathrm{d}u. \tag{1}$$

收稿日期: 2020-06-01　　　**基金项目:** 国家自然科学基金资助项目(11801556)

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其中在$\{W>0\}$条件下,ω表示几乎处处收敛的变量$W:=\lim_{n\uparrow\infty}Z_n/m^n$的连续密度函数,$\Phi$是标准正态分布函数,可表示为$\Phi(y):=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y}\mathrm{e}^{-\frac{z^2}{2}}\mathrm{d}z,\ y\in\mathbf{R}.$

Galton-Watson过程是一类特殊的Markov链,它被广泛应用在算法、数据结构、生物学、物理学等领域.有学者研究发现,在生物学研究中比值$\frac{Z_{n+1}}{Z_n}$可以对定量的聚合酶链反应实验的扩散速率、初始值大小等重要参数进行估计.例如:文献[1]对齐次分支过程的后代均值进行了估计.文献[2]通过定义灭绝概率q的一个估计序列$p_j(j=0,1,2,\cdots)$,证明了在$\alpha=f'(1-)\geqslant1$时估计序列$p_j(j=0,1,2,\cdots)$具有一致性和渐近正态性.文献[3]在θ-迁移过程为上临界Galton-Watson过程的假设下,得到了具有迁移情况的上临界Galton-Watson过程的渐近性以及对m和σ^2的估计结果.文献[4]对上临界的Galton-Watson过程的后代参数进行了估计,同时考虑了大偏差和下偏差对其估计值的影响.文献[5]对突变的聚合酶链反应模型中的类似S_{Z_n}的和$\sum_{x=1}^{S_0}u^{s(x)}$进行了研究,结果显示突变的聚合酶链反应模型具有很强的平均效应,即使对非常小的初始种群也是如此.受文献[1]和文献[2]的启发,本文考虑下述模型:令$X=(X_n)_{n\geqslant1}$表示独立同分布的(实值)随机变量族,且与Z独立;每一个X_n的均值都为0,方差都有限.令$S_n:=X_1+X_2+\cdots+X_n,\ n\geqslant0$,则在$\{Z_n>0\}$条件下,随机变量$R_n:=\frac{S_{Z_n}}{Z_n}$是有意义的.

依照文献[5]中的方法,很容易验证R_n的正态大偏差概率为

$$\lim_{n\uparrow\infty}\mathrm{P}(m^{n/2}R_n<x)=\int_0^x\Phi\left(\frac{xu^{1/2}}{\sigma}\right)\omega(u)\mathrm{d}u,\ x\in\mathbf{R},\tag{2}$$

其中$\sigma:=[\mathrm{E}(X_1^2)]^{1/2}$.令$\varepsilon_n>0$,考虑$\mathrm{P}(R_n\geqslant\varepsilon_n)$的相关问题.当$\varepsilon_n m^{n/2}\to\infty$时,则由式(2)知$R_n$具有简单的大偏差概率,$\lim_{n\uparrow\infty}\mathrm{P}(R_n\geqslant\varepsilon_n)=0$.

本文在更一般的情况下(当$n\uparrow\infty$时,$\varepsilon_n\to0$)利用Z的下偏差概率的结果来研究$R_n\geqslant\varepsilon_n$的概率收敛速率(大偏差),并且在文献[4]研究结果的基础上考虑在临界Galton-Watson过程下$\mathrm{P}(n^{1/2}R_n<x\mid Z_n>0)$的概率极限和在$\varepsilon_n\to0,\varepsilon_n^2n\to\infty$条件下$\varepsilon_n^2n\mathrm{P}(R_n\geqslant\varepsilon_n|Z_n>0)$的收敛速率,并给出其证明.本文所得结论可为估计突变聚合酶链反应实验中的初始粒子、扩散速率等参数提供理论基础.

1 相关假设及引理

1.1 基本假设

假设$\mathrm{E}(X_1^2)<\infty,\mathrm{E}(Z_1\log Z_1)<\infty$,并考虑$\varepsilon_n>0,\ \varepsilon_n\to0$和当$n\uparrow\infty$时,$\varepsilon_n^2n\to\infty$的情况.

1.2 相关结果

引理1[4](Fuk-Nagaev不等式) 对$k\geqslant1,\ \varepsilon_n>0,\ n\geqslant1,\ r>1$,有

$$\mathrm{P}(S_k\geqslant\varepsilon_nk)\leqslant k\mathrm{P}(X_1\geqslant r^{-1}\varepsilon_nk)+(\mathrm{e}r\sigma^2)^r\varepsilon_n^{-2r}k^{-r}.$$

引理2[6](基本引理) 如果$m=\mathrm{E}Z_1=1,\ \xi^2=\mathrm{Var}Z_1<\infty$,那么在$0\leqslant t<1$上一致有

$$\lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{1-f_n(t)}-\frac{1}{1-t}\right]=\frac{\xi^2}{2},$$

其中$f_n(t)$为矩母函数$f(t)$的n次迭代.

注1 如果$m=1,\ \xi^2<\infty$,那么当$n\to\infty$时有$1-f_n(0)=\mathrm{P}\{Z_n>0\}\sim\frac{2}{\xi^2n}\triangleq cn^{-1}$($c>0$为常数).

定理1[6] 如果$m=1,\ \xi^2<\infty$,那么有$\lim_{n\uparrow\infty}\mathrm{P}\left(\frac{Z_n}{n}>z\,|\,Z_n>0\right)=\exp\left\{-\frac{2z}{\xi^2}\right\},\ z>0.$

定理 2[7]　令 $\xi^2<\infty$,k 和 n 的比值 k/n 以有界的方式趋于无穷,那么有

$$\lim_{n\to\infty}\frac{\xi^4 n^2}{4d}\left(1+\frac{2d}{\xi^2 n}\right)^{k+1}\mathrm{P}(Z_n=kd)=1,$$

其中 d 为 $\{p_k;k\geqslant 0\}$ 的最大公约数.

引理 3　对所有 $n,\ k\geqslant 1$,存在一个常数 C_1,使得 $\mathrm{P}(Z_n=k\mid Z_n>0)\leqslant\dfrac{C_1}{n}$.

证明　令 $t=0$,则根据引理 2 有 $\lim\limits_{n\to\infty}\dfrac{1}{n}\left[\dfrac{1}{1-f_n(0)}-1\right]=\dfrac{\xi^2}{2}$,即 $\lim\limits_{n\to\infty}\dfrac{1}{n\mathrm{P}\{Z_n>0\}}=\dfrac{\xi^2}{2}$. 根据文献[7]中的性质知,若 $\xi^2<\infty$,则存在常数 c_1 使得 $\forall k,\ n\geqslant 1$,$\sup\limits_{n,k\geqslant 1}n^2\mathrm{P}(Z_n=k)\leqslant c_1$,因此有 $\mathrm{P}(Z_n=k)\leqslant\dfrac{C_2}{n^2}$. 所以当 n 足够大时有 $\mathrm{P}(Z_n=k\mid Z_n>0)=\dfrac{n\mathrm{P}(Z_n=k)}{n\mathrm{P}(Z_n>0)}\leqslant\dfrac{n\dfrac{C_2}{n^2}}{n\mathrm{P}(Z_n>0)}\leqslant\dfrac{C_2\xi^2}{n}\triangleq\dfrac{C_1}{n}$. 引理证毕.

2　主要结论及其证明

定理 3　假设 Z 是临界 Galton-Watson 过程,$m=1$,$\xi^2=\mathrm{Var}\,Z_1\in(0,\infty)$,则根据式(2)有

$$\lim_{n\uparrow\infty}\mathrm{P}(n^{1/2}R_n<x\mid Z_n>0)=\frac{2}{\xi^2}\int_0^\infty\Phi\left(\frac{xu^{1/2}}{\sigma}\right)\mathrm{e}^{-\frac{2z}{\xi^2}}\mathrm{d}u.\tag{3}$$

证明　根据全概率公式知,对 $\forall\delta\in(0,1)$ 和 $1<A<\infty$ 有

$$\mathrm{P}(n^{1/2}R_n<x\mid Z_n>0)=\sum_{k=1}^{\infty}\mathrm{P}(n^{1/2}S_k<xk)\,\mathrm{P}(Z_n=k\mid Z_n>0)=$$
$$\sum_{k<\delta n}\mathrm{P}(n^{1/2}S_k<xk)\,\mathrm{P}(Z_n=k\mid Z_n>0)+\sum_{\delta n\leqslant k\leqslant An}\mathrm{P}(n^{1/2}S_k<xk)\,\mathrm{P}(Z_n=k\mid Z_n>0)+$$
$$\sum_{k>An}\mathrm{P}(n^{1/2}S_k<xk)\,\mathrm{P}(Z_n=k\mid Z_n>0)\triangleq a+b+c.$$

以下分 3 部分证明定理 3.

(a) 根据引理 3,对 $\forall k,\ n\geqslant 1$,有

$$\sum_{k<\delta n}\mathrm{P}(n^{1/2}S_k<xk)\,\mathrm{P}(Z_n=k\mid Z_n>0)\leqslant\sum_{k<\delta n}\mathrm{P}(Z_n=k\mid Z_n>0)\leqslant\sum_{k<\delta n}\frac{C_3}{n}=C_3\delta.\tag{4}$$

(b) 当 $\delta n\leqslant k\leqslant An$,且当 n 足够大时,令 $d=1$,则根据定理 2 可知 $\mathrm{P}(Z_n=k)=(1+o(1))\cdot\dfrac{4}{\xi^4n^2}\left(1+\dfrac{2}{\xi^2 n}\right)^{-(k+1)}$. 再由引理 2 可得 $\lim\limits_{n\to\infty}\dfrac{1}{n\mathrm{P}(Z_n>0)}=\dfrac{\xi^2}{2}$. 所以,当 $\delta n\leqslant k\leqslant An$,且 n 足够大时有 $\mathrm{P}(Z_n=k\mid Z_n>0)=(1+o(1))\dfrac{1}{n}\dfrac{2}{\xi^2}\mathrm{e}^{-\frac{2}{\xi^2}\left(\frac{k}{n}\right)}$. 进而可得

$$\sum_{\delta n\leqslant k\leqslant An}\mathrm{P}(n^{1/2}S_k<xk)\,\mathrm{P}(Z_n=k\mid Z_n>0)=(1+o(1))\sum_{\delta n\leqslant k\leqslant An}\mathrm{P}\left(\frac{S_k}{\sigma\sqrt{k}}<\frac{x\sqrt{k}}{\sigma\sqrt{n}}\right)\frac{1}{n}\frac{2}{\xi^2}\mathrm{e}^{-\frac{2}{\xi^2}\left(\frac{k}{n}\right)}=$$
$$(1+o(1))\sum_{\delta n\leqslant k\leqslant An}\Phi\left(\frac{x\sqrt{k}}{\sigma\sqrt{n}}\right)\frac{1}{n}\frac{2}{\xi^2}\mathrm{e}^{-\frac{2}{\xi^2}\left(\frac{k}{n}\right)}=(1+o(1))\int_\delta^A\Phi\left(\frac{x\sqrt{u}}{\sigma}\right)\frac{2}{\xi^2}\mathrm{e}^{-\frac{2}{\xi^2}u}\mathrm{d}u=$$
$$(1+o(1))\frac{2}{\xi^2}\int_\delta^A\Phi\left(\frac{x\sqrt{u}}{\sigma}\right)\mathrm{e}^{-\frac{2}{\xi^2}u}\mathrm{d}u.\tag{5}$$

(c) $$\sum_{k>An}\mathrm{P}(n^{1/2}S_k<xk)\,\mathrm{P}(Z_n=k\mid Z_n>0)\leqslant\sum_{k>An}\mathrm{P}(Z_n=k\mid Z_n>0)=$$
$$1-\sum_{k<An}\mathrm{P}(Z_n=k\mid Z_n>0).\tag{6}$$

因 $\sum\limits_{k<An}\mathrm{P}(Z_n=k\mid Z_n>0)=1$,所以当 n 足够大时,式(6)等于 $o(1)$. 当 $A\to\infty$ 时,式(6)的极限趋于 0.

当 $\delta \to 0^+$ 时,式(4)的极限趋于 0. 当 $\delta \to 0^+$, $A \to \infty$ 时,

$$\sum_{\delta n \leqslant k \leqslant An} \mathrm{P}(n^{1/2} S_k < xk)\, \mathrm{P}(Z_n = k \mid Z_n > 0) \to \frac{2}{\xi^2} \int_\delta^A \Phi\left(\frac{x\sqrt{u}}{\sigma}\right) \mathrm{e}^{-\frac{2}{\xi^2}u} \mathrm{d}u.$$

综上所述,定理 3 证毕.

定理 4 假设 $\varepsilon_n \to 0$, $\varepsilon_n^2 n \to \infty$, 则 $\lim\limits_{n\uparrow\infty} \varepsilon_n^2 n \mathrm{P}(R_n \geqslant \varepsilon_n \mid Z_n > 0) = \dfrac{\sigma^2}{\xi^2}$.

证明 根据全概率公式知,对 $\forall \delta \in (0,1)$ 和 $1 < A < \infty$ 有

$$\varepsilon_n^2 n \mathrm{P}(R_n \geqslant \varepsilon_n \mid Z_n > 0) = \varepsilon_n^2 n \sum_{k=1}^{\infty} \mathrm{P}\Big(\sum_{i=1}^{k} X_i \geqslant \varepsilon_n k, Z_n = k \mid Z_n > 0\Big) =$$

$$\varepsilon_n^2 n \sum_{k=1}^{\infty} \mathrm{P}\Big(\sum_{i=1}^{k} X_i \geqslant \varepsilon_n k \mid Z_n = k\Big) \mathrm{P}(Z_n = k \mid Z_n > 0) =$$

$$\varepsilon_n^2 n \sum_{k < \delta/\varepsilon_n^2} \mathrm{P}\Big(\sum_{i=1}^{k} X_i \geqslant \varepsilon_n k \mid Z_n = k\Big) \mathrm{P}(Z_n = k \mid Z_n > 0) +$$

$$\varepsilon_n^2 n \sum_{\delta/\varepsilon_n^2 < k < A/\varepsilon_n^2} \mathrm{P}\Big(\sum_{i=1}^{k} X_i \geqslant \varepsilon_n k \mid Z_n = k\Big) \mathrm{P}(Z_n = k \mid Z_n > 0) +$$

$$\varepsilon_n^2 n \sum_{k > A/\varepsilon_n^2} \mathrm{P}\Big(\sum_{i=1}^{k} X_i \geqslant \varepsilon_n k \mid Z_n = k\Big) \mathrm{P}(Z_n = k \mid Z_n > 0) \triangleq \mathrm{I} + \mathrm{II} + \mathrm{III}.$$

下面分 3 部分证明定理 4.

(I) $$\lim_{n\uparrow\infty} \varepsilon_n^2 n \sum_{k < \delta/\varepsilon_n^2} \mathrm{P}\Big(\sum_{i=1}^{k} X_i \geqslant \varepsilon_n k \mid Z_n = k\Big) \mathrm{P}(Z_n = k \mid Z_n > 0) \leqslant$$

$$\lim_{n\uparrow\infty} \varepsilon_n^2 n \sum_{k < \delta/\varepsilon_n^2} \mathrm{P}(Z_n = k \mid Z_n > 0) = \lim_{n\uparrow\infty} (1 + 0(1)) \varepsilon_n^2 n \sum_{k < \delta/\varepsilon_n^2} \mathrm{P}(Z_n = k) \frac{\xi^2 n}{2} =$$

$$\lim_{n\uparrow\infty} (1 + 0(1)) \frac{\xi^2 n^2 \varepsilon_n^2}{2} \sum_{k < \delta/\varepsilon_n^2} \frac{4}{\xi^4 n^2} \left(1 + \frac{2}{\xi^2 n}\right)^{-(k+1)} = (1 + 0(1)) \frac{2\varepsilon_n^2}{\xi^2} \sum_{k < \delta/\varepsilon_n^2} (\mathrm{e}^{2/\xi^2})^{-\frac{k+1}{n}} \leqslant$$

$$(1 + 0(1)) \frac{2\varepsilon_n^2}{\xi^2} \sum_{k < \delta/\varepsilon_n^2} C_4^{-\frac{k}{n}} \leqslant (1 + 0(1)) \frac{2\varepsilon_n^2}{\xi^2} \frac{\delta}{\varepsilon_n^2} = \frac{2\delta}{\xi^2} \to 0.$$

(II) $$\lim_{n\uparrow\infty} \varepsilon_n^2 n \sum_{\delta/\varepsilon_n^2 < k < A/\varepsilon_n^2} \mathrm{P}\Big(\sum_{i=1}^{k} X_i \geqslant \varepsilon_n k \mid Z_n = k\Big) \mathrm{P}(Z_n = k \mid Z_n > 0) =$$

$$\varepsilon_n^2 n \sum_{\delta/\varepsilon_n^2 < k < A/\varepsilon_n^2} \lim_{n\uparrow\infty} \mathrm{P}\left(\frac{S_k}{\sigma\sqrt{k}} < \frac{\varepsilon_n k}{\sigma\sqrt{k}}\right) \mathrm{P}(Z_n = k \mid Z_n > 0) =$$

$$\varepsilon_n^2 n \sum_{\delta/\varepsilon_n^2 < k < A/\varepsilon_n^2} \bar{\Phi}\left(\frac{\varepsilon_n\sqrt{k}}{\sigma}\right) \mathrm{P}(Z_n = k \mid Z_n > 0) =$$

$$(1 + o(1))\, \varepsilon_n^2 n \sum_{\delta/\varepsilon_n^2 < k < A/\varepsilon_n^2} \bar{\Phi}\left(\frac{\varepsilon_n\sqrt{k}}{\sigma}\right) \frac{1}{n} \frac{2}{\xi^2} \mathrm{e}^{-\frac{2}{\xi^2}\left(\frac{k}{n}\right)} = (1 + o(1)) \frac{2}{\xi^2} \int_{\delta/\varepsilon_n^2}^{A/\varepsilon_n^2} \varepsilon_n^2 \bar{\Phi}\left(\frac{\varepsilon_n\sqrt{u}}{\sigma}\right) \mathrm{d}u =$$

$$(1 + o(1)) \frac{2}{\xi^2} \int_\delta^A \bar{\Phi}\left(\frac{\sqrt{x}}{\sigma}\right) \mathrm{d}x \leqslant \frac{2}{\xi^2} \int_0^\infty \bar{\Phi}\left(\frac{\sqrt{x}}{\sigma}\right) \mathrm{d}x = \frac{2}{\xi^2} \frac{1}{2} \sigma^2 = \frac{\sigma^2}{\xi^2},$$

其中 $\bar{\Phi}\left(\dfrac{\sqrt{x}}{\sigma}\right) = 1 - \Phi\left(\dfrac{\sqrt{x}}{\sigma}\right)$.

(III) 根据引理 1,令 $r = 2$,则有 $\mathrm{P}(S_k \geqslant \varepsilon_n k) \leqslant k\mathrm{P}\left(X_1 \geqslant \dfrac{\varepsilon_n k}{2}\right) + (2\mathrm{e}\sigma^2)^2 \varepsilon_n^{-4} k^{-2}$. 再结合引理 3 可得

$$\varepsilon_n^2 n \sum_{k > A/\varepsilon_n^2} \mathrm{P}\Big(\sum_{i=1}^{k} X_i \geqslant \varepsilon_n k \mid Z_n = k\Big) \mathrm{P}(Z_n = k \mid Z_n > 0) \leqslant \frac{C_5}{n} \varepsilon_n^2 n \sum_{k > A/\varepsilon_n^2} \mathrm{P}(Z_n = k \mid Z_n > 0) \leqslant$$

$$C_5\varepsilon_n^2\sum_{k>A/\varepsilon_n^2}\left[k\mathrm{P}\left(X_1\geqslant\frac{\varepsilon_n k}{2}\right)+(2\mathrm{e}\sigma^2)^2\varepsilon_n^{-4}k^{-2}\right]\leqslant$$

$$C_5\varepsilon_n^2\sum_{k>A/\varepsilon_n^2}\left[k\mathrm{P}\left(X_1\geqslant\frac{\varepsilon_n k}{2}\right)\right]+C_6\varepsilon_n^2\sum_{k>A/\varepsilon_n^2}\varepsilon_n^{-4}k^{-2}\triangleq \mathrm{D}+\mathrm{E}.$$

根据文献[4]中的式(76)，令 $\alpha=1$，则有 $\sum_{k>A/\varepsilon_n^2}\varepsilon_n^{-4}k^{-2}\leqslant\frac{C_7}{A}\varepsilon_n^{-2}$，$n>0$，$\varepsilon_n>0$，$A\geqslant 1$. 从而 D 部分变为

$$C_6\varepsilon_n^2\sum_{k>A/\varepsilon_n^2}\varepsilon_n^{-4}k^{-2}\leqslant C_6\varepsilon_n^2\frac{C_7}{A}\varepsilon_n^{-2}\leqslant\frac{C_8}{A}\to 0,\ 1<A\to\infty.$$

对 $k\geqslant 2$，根据 $\frac{k}{2}\leqslant k-1$ 可得 $(k-1)\,\mathrm{P}\left(X_1\geqslant\frac{\varepsilon_n k}{2}\right)\leqslant\int_{k-1}^{k}(x+1)\,\mathrm{P}\left(X_1\geqslant\frac{\varepsilon_n x}{2}\right)\mathrm{d}x$ 是连续的，从而有

$$C_5\varepsilon_n^2\sum_{k>A/\varepsilon_n^2}\left[k\mathrm{P}\left(X_1\geqslant\frac{\varepsilon_n k}{2}\right)\right]\leqslant C_5\varepsilon_n^2\sum_{k>A/\varepsilon_n^2}\int_{k-1}^{k}(x+2)\,\mathrm{P}\left(X_1\geqslant\frac{\varepsilon_n x}{2}\right)\mathrm{d}x=$$

$$C_5\varepsilon_n^2\int_{A/\varepsilon_n^2-1}^{\infty}(x+2)\,\mathrm{P}\left(X_1\geqslant\frac{\varepsilon_n x}{2}\right)\mathrm{d}x=$$

$$C_5\varepsilon_n^2\int_{A/\varepsilon_n^2-1}^{\infty}x\mathrm{P}\left(X_1\geqslant\frac{\varepsilon_n x}{2}\right)\mathrm{d}x+C_5\varepsilon_n^2\int_{A/\varepsilon_n^2-1}^{\infty}2\mathrm{P}\left(X_1\geqslant\frac{\varepsilon_n x}{2}\right)\mathrm{d}x=$$

$$C_5\varepsilon_n^2\int_{A/\varepsilon_n^2-1}^{\infty}x\mathrm{P}\left(X_1\geqslant\frac{\varepsilon_n x}{2}\right)\mathrm{d}x+(1+o(1))\,C_5\varepsilon_n^2\int_{A/\varepsilon_n^2-1}^{\infty}2\,\frac{2}{\varepsilon_n^2x^2}\mathrm{d}x=$$

$$C_5\varepsilon_n^2\int_{A/\varepsilon_n^2-1}^{\infty}x\mathrm{P}\left(X_1\geqslant\frac{\varepsilon_n x}{2}\right)\mathrm{d}x+o(1).$$

在上述式子中令 $v=\frac{x\varepsilon_n}{2}$，则可得

$$C_5\varepsilon_n^2\sum_{k>A/\varepsilon_n^2}\left[k\mathrm{P}\left(X_1\geqslant\frac{\varepsilon_n k}{2}\right)\right]\leqslant(1+o(1))\,C_5\varepsilon_n^2\int_{\frac{A}{2\varepsilon_n}-\frac{\varepsilon_n}{2}}^{\infty}\frac{\varepsilon_n}{2}v\,\frac{2}{\varepsilon_n}\mathrm{P}(X_1\geqslant v)\,\mathrm{d}v=$$

$$(1+o(1))\,C_5\varepsilon_n^2\int_{\frac{A}{2\varepsilon_n}-\frac{\varepsilon_n}{2}}^{\infty}v\mathrm{P}(X_1\geqslant v)\,\mathrm{d}v.$$

又由于本文假设二阶矩有限，因此当 $\frac{A}{2\varepsilon_n}-\frac{\varepsilon_n}{2}\to\infty$ 时，上述式子等于 $o(1)$. 进而有

$$\varepsilon_n^2 n\sum_{k>A/\varepsilon_n^2}\mathrm{P}\Big(\sum_{i=1}^{k}X_i\geqslant\varepsilon_n k\,|\,Z_n=k\Big)\,\mathrm{P}(Z_n=k\,|\,Z_n>0)=o(1).$$

综上所述，$\lim\limits_{n\uparrow\infty}\varepsilon_n^2 n\mathrm{P}(R_n\geqslant\varepsilon_n\,|\,Z_n>0)=0+o(1)+\frac{\sigma^2}{\xi^2}=\frac{\sigma^2}{\xi^2}$. 定理 4 证毕.

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（下转第 155 页）

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