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# 带有积分边界条件的 $p$ -Laplacian 分数阶微分方程边值问题正解的存在性

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**摘要:** 研究一类带有  $p$ -Laplacian 算子的分数阶微分方程的边值问题. 首先给出了边值问题解的表达式, 并分析了表达式中的格林函数的性质; 然后利用锥上的 Guo-Krasnosel'skii 不动点定理证明了该边值问题正解的存在性.

**关键词:** 正解;  $p$ -Laplacian; 格林函数

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## Existence of positive solutions for fractional differential equation involving integral boundary conditions with $p$ -Laplacian operator

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**Abstract:** In this paper, the fractional boundary value problem with  $p$ -Laplacian operator is discussed. Firstly, the expression of the solution of the boundary value problem is given, the properties of Green's function in the expression of solution are analyzed. Then, by using the Guo-Krasnosel'skii fixed-point theorem on the cone, we obtain the existence of the positive solution of the boundary value problem.

**Keywords:** positive solution;  $p$ -Laplacian; Green's function

### 0 引言

近年来分数阶微分方程在物理学、化学、控制系统等领域得到广泛应用, 其相关研究也得到一些进展<sup>[1-9]</sup>. 例如: 2012 年, Dong 等<sup>[6]</sup> 研究了如下的边值问题:

$$\begin{cases} D^\alpha(\phi_p(D^\alpha u(t))) = f(t, u(t)), & t \in [0, 1], \\ u(0) = u(1) = D^\alpha u(0) = D^\alpha u(1) = 0, \end{cases}$$

作者应用锥的伸拉缩的定理得到了该边值问题解的存在性. 2015 年, Han 等<sup>[7]</sup> 研究了如下的带有广义  $p$ -Laplacian 算子的边值问题:

$$\begin{cases} D_{0+}^\beta(\phi(D_{0+}^\alpha u(t))) = \lambda f(u(t)), & 0 < t < 1, \\ u(0) = u'(0) = u'(1) = 0, \\ \phi(D_{0+}^\alpha u(0)) = (\phi(D_{0+}^\alpha u(0)))' = 0, \end{cases}$$

其中  $1 < \alpha, \beta \leq 2$ , 作者利用锥上的 Guo-Krasnosel'skii 不动点定理得到了该边值问题正解的存在性. 2016 年, Günendi 和 Yaslan<sup>[8]</sup> 利用 Avery-Henderson 不动点定理和 Leggett-Williams 不动点定理, 得

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到了带积分边界条件的分数阶微分方程

$$\begin{cases} -D_{0+}^{\eta+2}(u''(t)) + f(u(t)) = 0, t \in [0, 1], \\ u''(0) = u'''(0) = \cdots = u^{(n-2)}(1) = 0, u'''(1) = 0, u'''(1) = 0, \\ \alpha u(0) - \beta u'(0) = \sum_{p=1}^{m-2} a_p \int_0^{\xi_p} u(s) ds, \\ \gamma u(0) + \delta u'(1) = \sum_{p=1}^{m-2} b_p \int_0^{\xi_p} u(s) ds \end{cases}$$

的多重正解的存在性, 其中  $n-1 < \eta \leq n$ ,  $n \geq 3$ ,  $\alpha, \beta, \gamma, \delta > 0$ ,  $a_p, b_p \geq 0$ . 2017 年, Li<sup>[9]</sup> 研究了下列分数阶边值问题:

$$\begin{cases} D_{0+}^{\beta}[\phi_p({}^C D_{0+}^{\alpha} u(t))] + f(t, u(t)) = 0, t \in (0, 1), \\ [\phi_p({}^C D_{0+}^{\alpha} u(0))] = \phi_p({}^C D_{0+}^{\alpha} u(0)) = \phi_p({}^C D_{0+}^{\alpha} u(1)) = 0, \\ u''(0) = u'(1) = 0, \\ au(0) + bu'(0) = \int_0^1 g(t)u(t)dt, \end{cases}$$

其中  $2 < \alpha, \beta \leq 3$ , 作者应用 Avery-Henderson 不动点定理给出了解的存在性.

受上述文献的启发, 本文研究如下分数阶边值问题:

$$\begin{cases} {}^C D_{0+}^{\alpha}[\phi_p(D_{0+}^{\beta} u(t))] = f(t, u(t)), t \in [0, 1], \\ [\phi_p(D_{0+}^{\beta} u(0))] = [\phi_p(D_{0+}^{\beta} u(0))]'' = \phi_p(D_{0+}^{\beta} u(1)) = 0, \\ {}^C D_{0+}^{\beta-2} u(0) = u(0) = 0, \\ u(1) = \int_0^1 g(t)u(t)dt, \end{cases} \quad (1)$$

其中  $2 < \alpha, \beta \leq 3$ ,  $5 < \alpha + \beta \leq 6$ ,  $\phi_p = |u|^{p-2}u$ ,  $p > 1$ ,  ${}^C D_{0+}^{\alpha}$  是 Caputo 型分数阶导数,  $D_{0+}^{\beta}$  是 Riemann-Liouville 型分数阶导数.

本文假设如下条件对边值问题(1) 中的  $g(t), f(t, u)$  成立:

(H<sub>1</sub>)  $g(t) : [0, 1] \rightarrow [0, +\infty)$ ,  $g(t) \in L^1[0, 1]$ ,  $0 < \int_0^1 g(\tau)d\tau < 1$ ;

(H<sub>2</sub>)  $f(t, u) : [0, 1] \times (0, +\infty) \rightarrow (0, +\infty)$  是连续的.

## 1 预备知识

**定义 1** 定义函数  $y : (0, +\infty) \rightarrow \mathbf{R}$  的  $\alpha$  ( $\alpha > 0$ ) 阶分数积分为  $I_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds$ ,

其中  $\Gamma(\alpha) = \int_0^{+\infty} e^{-x} x^{\alpha-1} dx$ .

**定义 2** 定义连续函数  $y : (0, +\infty) \rightarrow \mathbf{R}$  的  $\alpha$  ( $\alpha > 0$ ) 阶 Riemann-Liouville 型分数阶导数为

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_0^t (t-s)^{n-\alpha-1} y(s) ds, \text{ 其中 } n = [\alpha] + 1.$$

**引理 1**<sup>[10]</sup> 令  $\alpha > 0$ , 假设  $u, D_{0+}^{\alpha} u \in L^1(0, 1)$ , 则  $I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n}$ ,  $C_i \in \mathbf{R}$ ,  $i = 1, 2, \cdots, n$ ,  $n = [\alpha] + 1$ .

**定义 3** 定义连续函数  $y : (0, +\infty) \rightarrow \mathbf{R}$  的  $\alpha$  ( $\alpha > 0$ ) 阶 Caputo 型分数阶导数为  $D_{0+}^{\alpha} y(t) =$

$$\frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} y^{(n)}(s) ds, \text{ 其中 } n = [\alpha] + 1.$$

**引理 2**<sup>[11]</sup> 令  $\alpha > 0$ , 假设  $u, {}^C D_{0+}^{\alpha} u \in L^1(0, 1)$ , 则  $I_{0+}^{\alpha} {}^C D_{0+}^{\alpha} u(t) = u(t) + C_0 + C_1 t + C_2 t^2 + \cdots + C_{n-1} t^{n-1}$ ,  $C_i \in \mathbf{R}$ ,  $i = 1, 2, \cdots, n$ ,  $n = [\alpha] + 1$ .

**引理 3**<sup>[12]</sup> 对于  $t^\beta$  的 Caputo 型的  $\alpha$  ( $\alpha \in (n-1, n)$ ) 阶导数为

$${}^c D_{0+}^{\alpha} t^{\beta} = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, & \beta > n-1; \\ 0, & \beta \in \{0, 1, \dots, n-1\}; \end{cases}$$

**引理 4**<sup>[12]</sup> 令  $\beta > \alpha > 0$ ,  $f \in L^1[a, b]$ , 对于  $t \in [a, b]$ , 有  ${}^c D_{0+}^{\alpha} I_{a+}^{\beta} f(t) = I_{a+}^{\beta-\alpha} f(t)$  成立.

## 2 主要结果及其证明

**引理 5** 边值问题(1) 的解为  $u(t) = \int_0^1 G(t, s)v(s)ds$ , 其中

$$G(t, s) =$$

$$\frac{1}{A} \begin{cases} t^{\beta-1}(1-s)^{\beta-1} - t^{\beta-1} \int_s^1 g(\tau)(\tau-s)^{\beta-1} d\tau - \left(1 - \int_0^1 g(\tau)\tau^{\beta-1} d\tau\right) (t-s)^{\beta-1}, & 0 \leq s \leq t \leq 1; \\ t^{\beta-1}(1-s)^{\beta-1} - t^{\beta-1} \int_s^1 g(\tau)(\tau-s)^{\beta-1} d\tau, & 0 \leq t \leq s \leq 1, \end{cases}$$

$$v(t) = \phi_q \left( \int_0^1 H(t, \tau) f(\tau, u(\tau)) d\tau \right), \quad A = 1 - \int_0^1 g(t) t^{\beta-1} dt,$$

$$H(t, \tau) = \frac{1}{\Gamma(\alpha)} \begin{cases} (1-\tau)^{\alpha-1} - (t-\tau)^{\alpha-1}, & 0 \leq \tau \leq t \leq 1; \\ (1-\tau)^{\alpha-1}, & 0 \leq t \leq \tau \leq 1; \end{cases}$$

**证明** 由方程(1) 及引理 2, 得:

$$I_{0+}^{\alpha} {}^c D_{0+}^{\alpha} [\phi_p(D_{0+}^{\beta} u(t))] = I_{0+}^{\alpha} f(t, u(t)),$$

$$\phi_p(D_{0+}^{\beta} u(t)) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, u(\tau)) d\tau + d_0 + d_1 t + d_2 t^2.$$

由  $[\phi_p(D_{0+}^{\beta} u(0))] = [\phi_p(D_{0+}^{\beta} u(0))]' = 0$ , 可得  $d_1 = d_2 = 0$ . 再根据  $\phi_p(D_{0+}^{\beta} u(1)) = 0$ , 有

$$d_0 = -\frac{1}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} f(\tau, u(\tau)) d\tau,$$

所以

$$\begin{aligned} \phi_p(D_{0+}^{\beta} u(t)) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, u(\tau)) d\tau - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} f(\tau, u(\tau)) d\tau = \\ &= -\int_0^1 H(t, \tau) f(\tau, u(\tau)) d\tau. \end{aligned}$$

令  $D_{0+}^{\beta} u(t) = -v(t)$ , 再由引理 1 有  $u(t) = -\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} v(s) ds - C_1 t^{\beta-1} - C_2 t^{\beta-2} - C_3 t^{\beta-3}$ . 根据边值条件  $u(0) = 0$ , 得  $C_3 = 0$ . 又由引理 3 和引理 4 可得

$${}^c D_{0+}^{\beta-2} u(t) = -{}^c D_{0+}^{\beta-2} I_{0+}^{\beta} y(t) - C_1 {}^c D_{0+}^{\beta-2} t^{\beta-1} - C_2 {}^c D_{0+}^{\beta-2} t^{\beta-2} = -I_{0+}^2 y(t) - C_1 \Gamma(\beta) t - C_2 \Gamma(\beta-1).$$

由  ${}^c D_{0+}^{\beta-2} u(0) = 0$ , 得  $C_2 = 0$ . 再由  $u(1) = \int_0^1 g(t) u(t) dt$ , 可得:

$$-\frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} v(s) ds - C_1 = -\frac{1}{\Gamma(\beta)} \int_0^1 g(t) \int_0^t (t-s)^{\beta-1} v(s) ds dt - C_1 \int_0^1 t^{\beta-1} g(t) dt,$$

$$C_1 = \frac{-\frac{1}{\Gamma(\beta)} \int_0^1 g(t) \int_0^t (t-s)^{\beta-1} v(s) ds dt + \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} v(s) ds}{-\left(1 - \int_0^1 t^{\beta-1} g(t) dt\right)},$$

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} v(s) ds - \\ &= \left( \frac{\frac{1}{\Gamma(\beta)} \int_0^1 g(t) \int_0^t (t-s)^{\beta-1} v(s) ds dt - \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} v(s) ds}{1 - \int_0^1 t^{\beta-1} g(t) dt} \right) t^{\beta-1} = \end{aligned}$$

$$\left[ -\frac{1}{\Gamma(\beta)} \left( 1 - \int_0^1 t^{\beta-1} g(t) dt \right) \int_0^t (t-s)^{\beta-1} v(s) ds - t^{\beta-1} \frac{1}{\Gamma(\beta)} \int_0^1 \left( \int_s^1 g(t) (t-s)^{\beta-1} dt \right) v(s) ds + \right. \\ \left. t^{\beta-1} \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} v(s) ds \right] \times \frac{1}{1 - \int_0^1 t^{\beta-1} g(t) dt} = \int_0^1 G(t,s) v(s) ds.$$

**定理 1** 函数  $G(t,s)$  和  $H(t,\tau)$  满足如下性质:

(i)  $H(t,\tau) \geq 0$ ,  $G(t,s) \geq 0$ ,  $\tau, t, s \in [0,1]$ .

(ii)  $\max_{0 \leq t \leq 1} H(t,\tau) = H(\tau,\tau)$ ,  $\min_{\frac{1}{4} \leq t \leq \frac{3}{4}} H(t,\tau) \geq M(\tau)H(\tau,\tau)$ ,  $\tau \in [0,1]$ , 其中  $M(\tau) = 1 - \left( \frac{3-\tau}{4-\tau} \right)^{\alpha-1}$ .

(iii) 存在  $l \in (0,1)$  及  $s^* \in (s,1)$ , 使得:  $\max_{0 \leq t \leq 1} G(t,s) = G(s^*,s)$ ,  $t, s, \tau \in [0,1]$ ;  $\min_{\frac{1}{4} \leq t \leq \frac{3}{4}} G(t,s) \geq$

$lG(s^*,s)$ ,  $t, s, \tau \in (0,1)$ .

**证明** (i) 显然  $H(t,\tau) \geq 0$ ,  $\tau, t \in [0,1]$  是成立的. 下面证明  $G(t,s) \geq 0$ ,  $t, s \in [0,1]$ . 当  $0 \leq t \leq s \leq 1$  时, 有

$$AG(t,s) = \frac{1}{\Gamma(\beta)} \left[ t^{\beta-1} (1-s)^{\beta-1} - t^{\beta-1} \int_s^1 g(\tau) (\tau-s)^{\beta-1} d\tau \right] \geq \\ \frac{1}{\Gamma(\beta)} \left[ t^{\beta-1} (1-s)^{\beta-1} - t^{\beta-1} (1-s)^{\beta-1} \int_s^1 g(\tau) d\tau \right] \geq \frac{1}{\Gamma(\beta)} t^{\beta-1} (1-s)^{\beta-1} \left[ 1 - \int_0^1 g(\tau) d\tau \right] \geq 0;$$

当  $0 \leq s \leq t \leq 1$  时, 有

$$AG(t,s) = \frac{1}{\Gamma(\beta)} \left[ t^{\beta-1} (1-s)^{\beta-1} - t^{\beta-1} \int_s^1 g(\tau) (\tau-s)^{\beta-1} d\tau - \left( 1 - \int_0^1 g(\tau) \tau^{\beta-1} d\tau \right) (t-s)^{\beta-1} \right] = \\ \frac{1}{\Gamma(\beta)} t^{\beta-1} \left\{ (1-s)^{\beta-1} - \int_s^1 g(\tau) (\tau-s)^{\beta-1} d\tau - \left( 1 - \frac{s}{t} \right)^{\beta-1} \left( 1 - \int_0^1 g(\tau) \tau^{\beta-1} d\tau \right) \right\} \geq \\ \frac{1}{\Gamma(\beta)} t^{\beta-1} \left\{ (1-s)^{\beta-1} - \int_s^1 g(\tau) (\tau-s)^{\beta-1} d\tau - (1-s)^{\beta-1} \left( 1 - \int_0^1 g(\tau) \tau^{\beta-1} d\tau \right) \right\} = \\ \frac{1}{\Gamma(\beta)} t^{\beta-1} \left[ \int_0^1 g(\tau) (\tau-ts)^{\beta-1} d\tau - \int_s^1 g(\tau) (\tau-s)^{\beta-1} d\tau \right] \geq \\ \frac{1}{\Gamma(\beta)} t^{\beta-1} \left[ \int_0^1 g(\tau) (\tau-s)^{\beta-1} d\tau - \int_s^1 g(\tau) (\tau-s)^{\beta-1} d\tau \right] \geq 0.$$

(ii) 当  $0 \leq t \leq \tau \leq 1$  时,  $\frac{\partial H(t,\tau)}{\partial t} = 0$ ; 当  $0 \leq \tau \leq t \leq 1$  时,  $\frac{\partial H(t,\tau)}{\partial t} = -(\alpha-1)(t-\tau)^{\alpha-2} \leq 0$ .

所以  $\max_{0 \leq t \leq 1} H(t,\tau) = H(\tau,\tau) = (1-\tau)^{\alpha-1}$ . 当  $0 \leq t \leq \tau \leq 1$  时,  $H(t,\tau) = (1-\tau)^{\alpha-1}$ ; 当  $0 \leq \tau \leq t \leq 1$

时,  $\min_{\frac{1}{4} \leq t \leq \frac{3}{4}} H(t,\tau) = (1-\tau)^{\alpha-1} \left[ 1 - \left( \frac{\frac{3}{4}-\tau}{1-\tau} \right)^{\alpha-1} \right]$ ; 因此(ii) 成立.

(iii) 当  $0 \leq s \leq t \leq 1$  时,

$$A \frac{\partial G(t,s)}{\partial t} = \frac{1}{\Gamma(\beta-1)} \left[ t^{\beta-2} (1-s)^{\beta-1} - t^{\beta-2} \int_s^1 g(\tau) (\tau-s)^{\beta-1} d\tau - (t-s)^{\beta-2} \left( 1 - \int_0^1 g(\tau) \tau^{\beta-1} d\tau \right) \right], \\ A \frac{\partial^2 G(t,s)}{\partial t^2} = \frac{1}{\Gamma(\beta-2)} \left[ t^{\beta-3} \left( (1-s)^{\beta-1} - \int_s^1 g(\tau) (\tau-s)^{\beta-1} d\tau \right) - (t-s)^{\beta-3} \left( 1 - \int_0^1 g(\tau) \tau^{\beta-1} d\tau \right) \right] \leq \\ \frac{1}{\Gamma(\beta-2)} (t-s)^{\beta-3} \left[ (1-s)^{\beta-1} - \int_s^1 g(\tau) (\tau-s)^{\beta-1} d\tau - \left( 1 - \int_0^1 g(\tau) \tau^{\beta-1} d\tau \right) \right].$$

记  $F(s) = (1-s)^{\beta-1} - \int_s^1 g(\tau) (\tau-s)^{\beta-1} d\tau - \left( 1 - \int_0^1 g(\tau) \tau^{\beta-1} d\tau \right)$ ,  $F'(s) = -(\beta-1)(1-s)^{\beta-2} + (\beta-1) \cdot \int_s^1 g(\tau) (\tau-s)^{\beta-2} d\tau \leq (\beta-1)(1-s)^{\beta-2} \left( -1 + \int_0^1 g(\tau) d\tau \right) \leq 0$ , 所以  $F(s)$  是单调递减的. 又因  $F(0) = 0$ ,

所以有  $F(s) \leq 0$ , 从而  $A \frac{\partial^2 G(t,s)}{\partial t^2} \leq 0$ ,  $A \frac{\partial G(t,s)}{\partial t}$  关于  $t$  是单调递减的,  $AG(t,s)$  关于  $t$  是上凸的. 当

$t=1$  时,有

$$A \frac{\partial G(t,s)}{\partial t} = \frac{1}{\Gamma(\beta-1)} \left[ (1-s)^{\beta-1} - \int_s^1 g(\tau)(\tau-s)^{\beta-1} d\tau - (1-s)^{\beta-2} \left( 1 - \int_0^1 g(\tau)\tau^{\beta-1} d\tau \right) \right].$$

令

$$h(s) = (1-s)^{\beta-1} - \int_s^1 g(\tau)(\tau-s)^{\beta-1} d\tau - (1-s)^{\beta-2} \left( 1 - \int_0^1 g(\tau)\tau^{\beta-1} d\tau \right), \quad (2)$$

则  $h(0)=0$ ,  $h(1)=0$ . 当  $s \in (0,1)$  时,有

$$h'(s) = -(\beta-1)(1-s)^{\beta-2} + (\beta-1) \int_s^1 g(\tau)(\tau-s)^{\beta-2} d\tau + (\beta-2)(1-s)^{\beta-3} \left( 1 - \int_0^1 g(\tau)\tau^{\beta-1} d\tau \right).$$

令  $h'(s)=0$ , 有

$$(1-s)^{\beta-2} \left( 1 - \int_0^1 g(\tau)\tau^{\beta-1} d\tau \right) = \frac{1}{\beta-2} \left[ (\beta-1)(1-s)^{\beta-1} - (\beta-1)(1-s) \int_s^1 g(\tau)(\tau-s)^{\beta-2} d\tau \right]. \quad (3)$$

将式(3)代入式(2)得

$$\begin{aligned} h(s) &= (1-s)^{\beta-1} - \int_s^1 g(\tau)(\tau-s)^{\beta-1} d\tau - \frac{1}{\beta-2} \left[ (\beta-1)(1-s)^{\beta-1} - \right. \\ &\quad \left. (\beta-1)(1-s) \int_s^1 g(\tau)(\tau-s)^{\beta-2} d\tau \right] = \left[ (\beta-2)(1-s)^{\beta-1} - (\beta-2) \int_s^1 g(\tau)(\tau-s)^{\beta-1} d\tau - \right. \\ &\quad \left. (\beta-1)(1-s)^{\beta-1} + (\beta-1)(1-s) \int_s^1 g(\tau)(\tau-s)^{\beta-2} d\tau \right] / (\beta-2) = \\ &\quad \frac{1}{\beta-2} \left[ - (1-s)^{\beta-1} + (1-s(\beta-1)) \int_s^1 g(\tau)(\tau-s)^{\beta-1} d\tau \right] \leqslant \\ &\quad \frac{1}{\beta-2} \left[ - (1-s)^{\beta-1} + \int_s^1 g(\tau)(\tau-s)^{\beta-1} d\tau \right] < 0. \end{aligned}$$

所以当  $t=1$  时,有  $\frac{\partial G(t,s)}{\partial t} < 0$ ; 而当  $t=s$  时,有  $\frac{\partial G(t,s)}{\partial t} \geqslant 0$ . 所以一定存在  $t=s^*$ , 使得  $\frac{\partial G(s^*,s)}{\partial t} = 0$ , 即:

$$s^{*\beta-2}(1-s)^{\beta-1} - s^{*\beta-2} \int_s^1 g(\tau)(\tau-s)^{\beta-1} d\tau - (s^*-s)^{\beta-2} \left( 1 - \int_0^1 g(\tau)\tau^{\beta-1} d\tau \right) = 0,$$

$$s^* = \frac{s}{1 - \left[ \frac{(1-s)^{\beta-1} - \int_s^1 g(\tau)(\tau-s)^{\beta-1} d\tau}{1 - \int_0^1 g(\tau)\tau^{\beta-1} d\tau} \right]^{\frac{\beta-2}{2}}} > \frac{s}{1 - (1-s)^{\frac{\beta-1}{\beta-2}}}.$$

当  $0 \leqslant t \leqslant s \leqslant 1$  时,显然  $A \frac{\partial G(t,s)}{\partial t} \geqslant 0$ , 所以  $\max_{0 \leqslant t \leqslant 1} G(t,s) = G(s^*,s)$ .

当  $t \leqslant s$  时,

$$\begin{aligned} \frac{\min_{\frac{1}{4} \leqslant t \leqslant \frac{3}{4}} G(t,s)}{G(s^*,s)} &= \frac{\min_{\frac{1}{4} \leqslant t \leqslant \frac{3}{4}} \frac{1}{\Gamma(\beta)} \left[ t^{\beta-1} (1-s)^{\beta-1} - t^{\beta-1} \int_s^1 (\tau-s)^{\beta-1} g(\tau) d\tau \right]}{G(s^*,s)} \geqslant \\ &\quad \frac{\frac{1}{\Gamma(\beta)} \left( \frac{1}{4} \right)^{\beta-1} \left[ (1-s)^{\beta-1} - \int_s^1 (\tau-s)^{\beta-1} g(\tau) d\tau \right]}{\frac{1}{\Gamma(\beta)} \left[ s^{*\beta-1} (1-s)^{\beta-1} - s^{*\beta-1} \int_s^1 (\tau-s)^{\beta-1} g(\tau) d\tau - \left( 1 - \int_0^1 \tau^{\beta-1} g(\tau) d\tau \right) (s^*-s)^{\beta-1} \right]} \geqslant \\ &\quad \frac{\left( \frac{1}{4} \right)^{\beta-1} \left[ (1-s)^{\beta-1} - \int_s^1 (\tau-s)^{\beta-1} g(\tau) d\tau \right]}{s^{*\beta-1} (1-s)^{\beta-1} - s^{*\beta-1} \int_s^1 (\tau-s)^{\beta-1} g(\tau) d\tau} = \left( \frac{1}{4s^*} \right)^{\beta-1} \geqslant \left( \frac{1}{4} \right)^{\beta-1}. \end{aligned}$$

当  $s \leq t$  时,

$$\frac{\min_{\frac{1}{4} \leq t \leq \frac{3}{4}} G(t, s)}{G(s^*, s)} = \frac{\min_{\frac{1}{4} \leq t \leq \frac{3}{4}} \left[ t^{\beta-1} (1-s)^{\beta-1} - t^{\beta-1} \int_s^1 g(\tau) (\tau-s)^{\beta-1} d\tau - \left( 1 - \int_0^1 g(\tau) \tau^{\beta-1} d\tau \right) (t-s)^{\beta-1} \right]}{s^{*\beta-1} (1-s)^{\beta-1} - s^{*\beta-1} \int_s^1 g(\tau) (\tau-s)^{\beta-1} d\tau - \left( 1 - \int_0^1 g(\tau) \tau^{\beta-1} d\tau \right) (s^*-s)^{\beta-1}}, \quad (4)$$

其中  $s \leq s^*$ . 下面分两种情况对式(4)进行讨论:

1) 当  $s \leq t \leq s^*$  时,  $A \frac{\partial G(t, s)}{\partial t} \geq 0$ , 所以当  $t = \frac{1}{4}$  时公式(4)可取得最小值, 即:

$$\begin{aligned} \frac{\min_{\frac{1}{4} \leq t \leq \frac{3}{4}} G(t, s)}{G(s^*, s)} &= \frac{\frac{1}{\Gamma(\beta)} \left[ \left( \frac{1}{4} \right)^{\beta-1} (1-s)^{\beta-1} - \left( \frac{1}{4} \right)^{\beta-1} \int_s^1 g(\tau) (\tau-s)^{\beta-1} d\tau - \left( 1 - \int_0^1 g(\tau) \tau^{\beta-1} d\tau \right) \left( \frac{1}{4} - s \right)^{\beta-1} \right]}{\frac{1}{\Gamma(\beta)} \left[ s^{*\beta-1} (1-s)^{\beta-1} - s^{*\beta-1} \int_s^1 g(\tau) (\tau-s)^{\beta-1} d\tau - \left( 1 - \int_0^1 g(\tau) \tau^{\beta-1} d\tau \right) (s^*-s)^{\beta-1} \right]} = \\ &= \left( \frac{1}{4s^*} \right)^{\beta-1} \frac{(1-s)^{\beta-1} - \int_s^1 g(\tau) (\tau-s)^{\beta-1} d\tau - \left( 1 - \int_0^1 g(\tau) \tau^{\beta-1} d\tau \right) (1-4s)^{\beta-1}}{(1-s)^{\beta-1} - \int_s^1 g(\tau) (\tau-s)^{\beta-1} d\tau - \left( 1 - \int_0^1 g(\tau) \tau^{\beta-1} d\tau \right) \left( 1 - \frac{s}{s^*} \right)^{\beta-1}} \geq \\ &= \left( \frac{1}{4s^*} \right)^{\beta-1} \frac{(1-s)^{\beta-1} - \int_s^1 g(\tau) (\tau-s)^{\beta-1} d\tau - \left( 1 - \int_0^1 g(\tau) \tau^{\beta-1} d\tau \right) (1-4s)^{\beta-1}}{(1-s)^{\beta-1} - \int_s^1 g(\tau) (\tau-s)^{\beta-1} d\tau - \left( 1 - \int_0^1 g(\tau) \tau^{\beta-1} d\tau \right) (1-4s)^{\beta-1}} = \left( \frac{1}{4} \right)^{\beta-1}. \end{aligned}$$

2) 当  $s \leq s^* \leq t$  时,  $A \frac{\partial G(t, s)}{\partial t} \leq 0$ , 所以当  $t = \frac{3}{4}$  时公式(4)可取得最小值, 即:

$$\begin{aligned} \frac{\min_{\frac{1}{4} \leq t \leq \frac{3}{4}} G(t, s)}{G(s^*, s)} &= \frac{\frac{1}{\Gamma(\beta)} \left[ \left( \frac{3}{4} \right)^{\beta-1} (1-s)^{\beta-1} - \left( \frac{3}{4} \right)^{\beta-1} \int_s^1 g(\tau) (\tau-s)^{\beta-1} d\tau - \left( 1 - \int_0^1 g(\tau) \tau^{\beta-1} d\tau \right) \left( \frac{3}{4} - s \right)^{\beta-1} \right]}{\frac{1}{\Gamma(\beta)} \left[ s^{*\beta-1} (1-s)^{\beta-1} - s^{*\beta-1} \int_s^1 g(\tau) (\tau-s)^{\beta-1} d\tau - \left( 1 - \int_0^1 g(\tau) \tau^{\beta-1} d\tau \right) (s^*-s)^{\beta-1} \right]} = \\ &= \left( \frac{3}{4s^*} \right)^{\beta-1} \frac{(1-s)^{\beta-1} - \int_s^1 g(\tau) (\tau-s)^{\beta-1} d\tau - \left( 1 - \int_0^1 g(\tau) \tau^{\beta-1} d\tau \right) \left( 1 - \frac{4}{3}s \right)^{\beta-1}}{(1-s)^{\beta-1} - \int_s^1 g(\tau) (\tau-s)^{\beta-1} d\tau - \left( 1 - \int_0^1 g(\tau) \tau^{\beta-1} d\tau \right) \left( 1 - \frac{s}{s^*} \right)^{\beta-1}} \geq \\ &= \left( \frac{1}{4} \right)^{\beta-1} \frac{(1-s)^{\beta-1} - \int_s^1 g(\tau) (\tau-s)^{\beta-1} d\tau - \left( 1 - \int_0^1 g(\tau) \tau^{\beta-1} d\tau \right) (1-s)^{\beta-1}}{(1-s)^{\beta-1} - \int_s^1 g(\tau) (\tau-s)^{\beta-1} d\tau - \left( 1 - \int_0^1 g(\tau) \tau^{\beta-1} d\tau \right) (1-s)^{\frac{(\beta-1)^2}{(\beta-2)}}}. \end{aligned}$$

$$\text{令 } T(s) = \frac{(1-s)^{\beta-1} - \int_s^1 g(\tau) (\tau-s)^{\beta-1} d\tau - \left( 1 - \int_0^1 g(\tau) \tau^{\beta-1} d\tau \right) (1-s)^{\beta-1}}{(1-s)^{\beta-1} - \int_s^1 g(\tau) (\tau-s)^{\beta-1} d\tau - \left( 1 - \int_0^1 g(\tau) \tau^{\beta-1} d\tau \right) (1-s)^{\frac{(\beta-1)^2}{(\beta-2)}}}. \text{ 因为}$$

$$\lim_{s \rightarrow 0^+} T(s) = \frac{\int_0^1 (1-\tau) \tau^{\beta-2} g(\tau) d\tau}{\int_0^1 (1-\tau) \tau^{\beta-2} g(\tau) d\tau + \frac{1}{\beta-2} \left( 1 - \int_0^1 g(\tau) \tau^{\beta-1} d\tau \right)} = \alpha > 0,$$

根据保号性可知存在  $\delta > 0$ , 使得当  $0 < s < \delta$  时, 有  $T(s) > \frac{\alpha}{2}$ , 因此

$$\frac{\min_{\frac{1}{4} \leq t \leq \frac{3}{4}} G(t, s)}{G(s^*, s)} \geq \frac{\alpha}{2} \left( \frac{1}{4} \right)^{\beta-1}, \quad 0 < s < \delta.$$

当  $\delta \leq s \leq s^*$  时, 有

$$\begin{aligned} \frac{\min_{\frac{1}{4} \leq t \leq \frac{3}{4}} G(t, s)}{G(s^*, s)} &= \left( \frac{3}{4s^*} \right)^{\beta-1} \frac{(1-s)^{\beta-1} - \int_s^1 g(\tau)(\tau-s)^{\beta-1} d\tau - \left(1 - \int_0^1 g(\tau)\tau^{\beta-1} d\tau\right) \left(1 - \frac{4}{3}s\right)^{\beta-1}}{(1-s)^{\beta-1} - \int_s^1 g(\tau)(\tau-s)^{\beta-1} d\tau - \left(1 - \int_0^1 g(\tau)\tau^{\beta-1} d\tau\right) \left(1 - \frac{s}{s^*}\right)^{\beta-1}} \geq \\ &\left( \frac{3}{4s^*} \right)^{\beta-1} \frac{(1-s)^{\beta-1} - \int_s^1 g(\tau)(\tau-s)^{\beta-1} d\tau - \left(1 - \int_0^1 g(\tau)\tau^{\beta-1} d\tau\right) \left(1 - \frac{4}{3}s\right)^{\beta-1}}{(1-s)^{\beta-1} - \int_s^1 g(\tau)(\tau-s)^{\beta-1} d\tau} \geq \\ &\left( \frac{1}{4} \right)^{\beta-1} \left[ 1 - \frac{1 - \int_0^1 g(\tau)\tau^{\beta-1} d\tau}{1 - \int_s^1 g(\tau) \left( \frac{\tau-s}{1-s} \right)^{\beta-1} d\tau} \right] \geq \left( \frac{1}{4} \right)^{\beta-1} \left[ 1 - \frac{1 - \int_0^1 g(\tau)\tau^{\beta-1} d\tau}{1 - \int_{\delta}^1 g(\tau)\tau^{\beta-1} d\tau} \right] = \gamma. \end{aligned}$$

当取  $l = \min \left\{ \left( \frac{1}{4} \right)^{\beta-1}, \frac{\alpha}{2} \left( \frac{1}{4} \right)^{\beta-1}, \gamma \right\}$  时, 定理 1 中 (iii) 成立.

**引理 6** 令  $X$  是一个 Banach 空间,  $P$  是  $X$  中的一个锥, 假设  $\Omega_1$  和  $\Omega_2$  是  $X$  中的开集, 且  $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$ . 设  $S: P \rightarrow P$  是一个完全连续的算子, 且满足下面的条件之一, 则  $S$  在  $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$  上有不动点.

(A1)  $\|Sw\| \leq \|w\|$ ,  $w \in P \cap \partial\Omega_1$ ,  $\|Sw\| \geq \|w\|$ ,  $w \in P \cap \partial\Omega_2$ ;

(A2)  $\|Sw\| \geq \|w\|$ ,  $w \in P \cap \partial\Omega_1$ ,  $\|Sw\| \leq \|w\|$ ,  $w \in P \cap \partial\Omega_2$ .

下面应用引理 6, 证明边值问题解的存在性.

在空间  $E \in C[0, 1]$  中定义最大模  $\|u\| = \max_{0 \leq t \leq 1} |u(t)|$ , 则  $E$  为 Banach 空间,  $P$  为  $E$  上的锥,

$$P = \{u \in E \mid u(t) \geq 0, \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} u(t) \geq l\|u\|\}, \quad l = \min \left\{ \left( \frac{1}{4} \right)^{\beta-1}, \frac{\alpha}{2} \left( \frac{1}{4} \right)^{\beta-1}, \gamma \right\}.$$

定义算子  $T: P \rightarrow P$  如下:

$$Tu(t) = \int_0^1 G(t, s) \phi_q \left( \int_0^1 H(s, \tau) f(\tau, u(\tau)) d\tau \right) ds. \quad (5)$$

**引理 7** 假设  $f(t, u)$  在  $[0, 1] \times [0, +\infty)$  上是连续的, 且条件  $(H_2)$  成立, 则式 (5) 定义的算子  $T: P \rightarrow P$  是完全连续算子.

**证明** 首先证明  $T: P \rightarrow P$ . 由条件  $(H_2)$  知  $Tu(t) \geq 0$ . 再由定理 1 中的 (iii), 有

$$\min_{\frac{1}{4} \leq t \leq \frac{3}{4}} (Tu(t)) = \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} \int_0^1 G(t, s) v(s) ds \geq l \int_0^1 G(s^*, s) v(s) ds = l\|Tu\|,$$

因此得到  $T: P \rightarrow P$ .

其次证明  $T$  是有界的. 令  $D \subset P$  且有界, 那么存在一个实数  $r$ , 当  $u \in D$  时, 有  $\|u\| \leq r$ . 令  $B =$

$\max_{t \in [0, 1], u \in D} |f(t, u)|$ , 可得

$$\begin{aligned} \|Tu(t)\| &= \left| \int_0^1 G(t, s) \phi_q \left( \int_0^1 H(s, \tau) f(\tau, u(\tau)) d\tau \right) ds \right| \leq \\ &\int_0^1 |G(t, s)| \phi_q \left( \int_0^1 |H(s, \tau)| |f(\tau, u(\tau))| d\tau \right) ds \leq \phi_q(B) \int_0^1 G(s^*, s) \phi_q \left( \int_0^1 H(s, \tau) d\tau \right) ds := M_1, \end{aligned}$$

所以  $T$  是有界的.

下面往证  $T(D)$  是等度连续的. 因为  $G(t, s)$  连续, 所以对于任意的  $\varepsilon > 0$ , 存在  $\eta > 0$ , 对任意的  $t_1,$

$t_2 \in [0, 1]$ ,  $|t_1 - t_2| < \eta$ , 有  $|G(t_2, s) - G(t_1, s)| < \frac{\varepsilon}{\phi_q(B) \int_0^1 \phi_q \left( \int_0^1 H(\tau, \tau) d\tau \right) ds}$ , 则有

$$\begin{aligned} |Tu(t_2) - Tu(t_1)| &= \left| \int_0^1 G(t_2, s)v(s)ds - \int_0^1 G(t_1, s)v(s)ds \right| \leq \\ &\int_0^1 |G(t_2, s) - G(t_1, s)| \phi_q \left( \int_0^1 H(\tau, \tau)f(\tau, u(\tau))d\tau \right) ds < \varepsilon, \end{aligned}$$

所以得到  $T$  是等度连续的. 再由 Arzela-Ascoli 定理可知,  $T$  是完全连续的.

**定理 2**  $f(t, u)$  在  $[0, 1] \times [0, +\infty)$  上是连续的. 假设存在两个不同的正数  $r_1, r_2$ , 使:

$$(i) f(t, u) \leq \phi_p(Mr_1), (t, u) \in [0, 1] \times [0, r_1], M = \left( \int_0^1 G(s^*, s)\phi_q \left( \int_0^1 H(\tau, \tau)d\tau \right) ds \right)^{-1},$$

$$(ii) f(t, u) \geq \phi_p(Nr_2), (t, u) \in [\frac{1}{4}, \frac{3}{4}] \times [0, r_2], N = \left( \int_{\frac{1}{4}}^{\frac{3}{4}} lG(s^*, s)\phi_q \left( \int_{\frac{1}{4}}^{\frac{3}{4}} M(\tau)H(\tau, \tau)d\tau \right) ds \right)^{-1},$$

则边值问题(1)至少有一个正解  $u$ , 且  $\min\{r_1, r_2\} \leq \|u\| \leq \max\{r_1, r_2\}$ .

**证明** 由引理 7 可知  $T: P \rightarrow P$  是完全连续的. 设  $0 < r_1 < r_2$ , 且  $\Omega_1 = \{u \in P, \|u\| < r_1\}$ ,  $\Omega_2 = \{u \in P, \|u\| < r_2\}$ . 下面分两步证明此定理成立.

1) 对于  $u \in \partial\Omega_1$ , 在  $t \in [0, 1]$  时, 有  $0 \leq u(t) \leq r_1$ , 且  $\|u\| = r_1$ . 再根据条件(i) 有  $\|Tu(t)\| = \max_{0 \leq t \leq 1} \left| \int_0^1 G(t, s)v(s)ds \right| \leq Mr_1 \int_0^1 G(s^*, s)\phi_q \left( \int_0^1 H(\tau, \tau)d\tau \right) ds = r_1 = \|u\|$ , 所以  $\|Tu\| \leq \|u\|$ ,  $u \in \partial\Omega_1$ .

2) 对于  $u \in \partial\Omega_2$ , 根据本文定义的锥  $P$ , 有  $u(t) \geq l\|u\|$ ,  $l \in (0, 1)$ ,  $t \in [\frac{1}{4}, \frac{3}{4}]$ , 且  $\|u\| = r_2$ . 再根据(ii) 有  $\|Tu(t)\| \geq \int_{\frac{1}{4}}^{\frac{3}{4}} G(t, s)v(s)ds \geq Nr_2 \int_{\frac{1}{4}}^{\frac{3}{4}} lG(s^*, s)\phi_q \left( \int_{\frac{1}{4}}^{\frac{3}{4}} M(\tau)H(\tau, \tau)f(\tau, u(\tau))d\tau \right) ds = r_2 = \|u\|$ , 所以  $\|Tu\| \geq \|u\|$ ,  $u \in \partial\Omega_2$ .

综上, 再根据引理 6 可知,  $T$  在  $P \cap (\overline{\Omega_2} \setminus \Omega_1)$  上有不动点, 所以边值问题(1)至少有一个正解.

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