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带有 Robin 边界条件的分数阶q-差分方程的 Lyapunov 型不等式

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摘要: 考虑带有 Robin 边界条件的分数阶 q-差分方程 ${}^{C}D_{q}^{\alpha}u(t)+X(t)u(t)=0\ (0<t<1)$ 所满足的 Lyapunov 型不等式. 首先利用 Robin 边界条件得到该方程解的表达式, 然后通过分析格林函数得到格林函数的估值, 进而得到了该方程相应的 Lyapunov 型不等式.

关键词: Robin 边界条件; 分数阶 q-差分; Lyapunov 型不等式
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Lyapunov type inequalities for fractional q-difference equations with Robin boundary conditions

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Abstract: We consider the Lyapunov type inequalities satisfied by the fractional q-difference equation ${}^{C}D_{q}^{\alpha}u(t)+X(t)u(t)=0\ (0<t<1)$ with Robin boundary conditions. The Robin boundary conditions are used to get the expressions of solution for the equation. By analyzing Green's function, we get the estimate of the Green's function. Further the corresponding Lyapunov type inequalities are obtained.

Keywords: Robin boundary conditions; fractional order q-difference; Lyapunov-type inequality

0 引言

Lyapunov 在文献[1]中提出了经典的 Lyapunov 型不等式, 即若微分方程 $u''(t)+X(t)u(t)=0$ ($X(t)$ 是$[a,b]$上的连续函数) 存在满足条件 $u(a)=u(b)=0$ 的非平凡解 $u(t)$, 则不等式 $(b-a)\cdot\int_a^b |X(t)|\,\mathrm{d}t>4$ 成立. 此后, 很多学者对该型不等式进行了研究. 例如: 2013 年, R. A. C. Ferreira 在文献[2]中, 根据 Riemann-Liouville 分数阶导数, 得出一个 Lyapunov 型不等式. 即对于边值问题:

$$\begin{cases} {}_{a}D^{\alpha}u(t)+X(t)u(t)=0,\ a<t<b,\ 1<\alpha\leqslant 2; \\ u(a)=u(b)=0, \end{cases}$$

其中 ${}_{a}D^{\alpha}$ 为阶数为 α 的 Riemann-Liouville 分数阶导数, 若该方程存在非平凡解 $u(t)$, 则有 $\int_a^b |X(t)|\,\mathrm{d}t>\Gamma(\alpha)\left(\frac{4}{b-a}\right)^{\alpha-1}$. 2014 年, R. A. C. Ferreira 在文献[3]中给出了另一类 Lyapunov 型不等式. 即对于 Captuo型分数阶边值问题:

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$$\begin{cases} {}_a^C D^\alpha u(t) + X(t)u(t) = 0,\ a < t < b,\ 1 < \alpha \leqslant 2; \\ u(a) = u(b) = 0, \end{cases}$$

其中${}_a^C D^\alpha$ 为次数为 α 的 Caputo 型分数阶导数,若该方程存在非平凡解 $u(t)$,则有$\int_a^b |X(t)|\mathrm{d}t > \frac{\Gamma(\alpha)\alpha^\alpha}{[(\alpha-1)(b-a)]^{\alpha-1}}$.受上述研究结果启发,本文讨论带有 Robin 边界条件的分数阶 q-差分方程:

$$\begin{cases} {}^C D_q^\alpha u(t) + X(t)u(t) = 0,\ 0 < t < 1; \\ u(0) - D_q u(0) = u(1) + D_q u(1) = 0. \end{cases} \tag{$*$}$$

这里 $1 < \alpha \leqslant 2$, $q \in [0,1)$, $f:[0,1] \to \mathbf{R}$ 是连续函数.

1　预备知识

定义 1[4]　f 在$[a,b]$上连续,$q \in [0,1)$,定义 $f(t)$ 一阶 q-差分为

$${}_a D_q f(t) = \frac{f(t) - f(qt + (1-q)a)}{(1-q)(t-a)},\ t \neq a.$$

其中${}_a D_q f(a) = \lim_{t\to a} {}_{t_k} D_q f(t)$.显然,如果 f 在(a,b)上可微,则 $\lim_{q\to 1^-} {}_a D_q f(t) = f'(t)$, $t \in (a,b)$.同样,0 阶 q-差分和 n 阶 q-差分定义为${}_a D_q^0 f(t) = f(t)$, ${}_a D_q^n f(t) = {}_a D_q({}_a D_q^{n-1} f)(t)$.

定义 2[5-6]　令 $q \in [0,1)$, $a \in \mathbf{R}$,对于 $r \in \mathbf{R}$,定义$[r]_q = \frac{1-q^r}{1-q}$.

定义 3[5-6]　当 $n \in \mathbf{N}:\{0,1,2,\cdots\}$, $x,y \in \mathbf{R}$,定义 q-类幂函数$(x-y)_a^{(n)}$ 为

$$(x-y)_a^{(n)} = \prod_{i=0}^{n-1}((x-a)-(y-a)q^i),\ n \in \mathbf{N},\ (x,y) \in \mathbf{R}^2.$$

一般地,当 $n \in \mathbf{R}$ 时,$(x-y)_a^{(n)} = (x-a)^n \prod_{i=0}^{\infty} \frac{(x-a)-q^i(y-a)}{(x-a)-q^{i+n}(y-a)}$.当 $n=0$ 时,$(x-y)_a^{(0)} = 1$.如果 $y=0$, $n>0$,那么 $x^{(n)} = x^n$,同时 $0^{(n)} = 0$.

定义 4[6]　f 在$[a,b]$上连续,对任意的 $\alpha > 0$,Riemann-Liouville 型的分数阶 q-积分定义为$({}_a I_q^\alpha f)(t) = \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-(qs+(1-q)a))_a^{(\alpha-1)} f(s)\,{}_a\mathrm{d}_q s$.规定$({}_0 I_q^0 f)(t) = f(t)$, $t \in [a,b]$.

性质 1　对于任意的 $t,s \in [a,b]$,下列等式成立:

1) $[a(t-s)]^{(\alpha)} = a^\alpha (t-s)^{(\alpha)}$.

2) ${}_t({}_a D_q (t-s)_a^{(\alpha)}) = [\alpha]_q (t-s)_a^{(\alpha-1)}$.

3) ${}_s({}_a D_q (t-s)_a^{(\alpha)}) = -[\alpha]_q (t-(qs+(1-q)a))_a^{(\alpha-1)}$.

4) $\left({}_x D_q \int_0^x f(x,t)\mathrm{d}_q t\right)(x) = \int_0^x {}_x D_q f(x,t)\mathrm{d}_q t + f(qx,x)$.

引理 1　已知 f 在$[a,b]$上连续,假设对于任意的 $q \in [0,1)$,有${}_a D_q f(t) \leqslant 0$ (${}_a D_q f(t) \geqslant 0$), $a < t \leqslant b$,则 f 是递增函数.

证明　令$(x,y) \in [a,b] \times [a,b]$且 $x < y$, $q = \frac{x-a}{y-a}$,则对于 $q \in [0,1)$,有

$${}_a D_q f(y) = \frac{f(y) - f(qy+(1-q)a)}{(1-q)(y-a)} = \frac{f(y) - f(x)}{y-x} \leqslant 0.$$

由此得出 $f(y) \leqslant f(x)$,引理 1 得证.

引理 2[7]　f 和 g 是$[a,b]$上的连续函数,则:

1) $f \leqslant g \Rightarrow \int_a^b f(s)\,{}_a\mathrm{d}_q s \leqslant \int_a^b g(s)\,{}_a\mathrm{d}_q s$,

2) $\left|\int_a^b f(s)\,{}_a\mathrm{d}_q s\right| \leqslant \int_a^b |f(s)|\,{}_a\mathrm{d}_q s$.

引理 3 当 $x\neq a$ 时,式 $(x-y)_a^{(n)}=(x-a)^n\left(1-\dfrac{y-a}{x-a}\right)^{(n)}$ 成立.

证明 因为 $\left(1-\dfrac{y-a}{x-a}\right)^{(n)}=\prod\limits_{i=0}^{\infty}\dfrac{1-q^i\dfrac{y-a}{x-a}}{1-q^{i+n}\dfrac{y-a}{x-a}}=\prod\limits_{i=0}^{\infty}\dfrac{(x-a)-q^i(y-a)}{(x-a)-q^{i+n}(y-a)}=\dfrac{(x-y)_a^{(n)}}{(x-a)^n}$,所以引理 3 得证.

引理 4[8] f 在 $[a,b]$ 上连续,p 为正整数,对于任意的 $\alpha\geqslant 0$,$[a,b]$ 上的 Caputo 型分数阶 q-差分满足 ${}_aI_q^{\alpha}\,{}_a^CD_q^{\alpha}f(t)=f(t)-\sum\limits_{i=0}^{\alpha-1}\dfrac{(t-a)^k}{\Gamma_q(k+1)}\,{}_aD_q^kf(a)$, $t\in[a,b]$.

2 主要结果及其证明

为方便引入如下记号:

$$h_1(t,s)=1+t-\frac{3(t-qs)^{(\alpha-1)}}{(1-qs)^{(\alpha-1)}+[\alpha-1]_q(1-qs)^{(\alpha-2)}},$$
$$h_2(t,s)=1+t. \tag{1}$$

引理 5 $u\in C[0,1]$ 是问题(*)的解,当且仅当 u 满足 $u(t)=\int_0^1G(t,s)X(s)u(s)\mathrm{d}_qs$,其中:

$$G(t,s)=\frac{(1-qs)^{(\alpha-2)}(1-q^{\alpha-1}s+[\alpha-1]_q)}{3\Gamma_q(\alpha)}H(t,s); \tag{2}$$

$$H(t,s)=\begin{cases}h_1(t,s),\ 0\leqslant s\leqslant t\leqslant 1;\\ h_2(t,s),\ 0\leqslant t\leqslant s\leqslant 1.\end{cases} \tag{3}$$

证明 由引理 4 及定义 4,得 $u(t)=-\int_0^t\dfrac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}X(s)u(s)\mathrm{d}_qs+C_1+C_2t$,其中 C_1 和 C_2 是常数.故有 $u(0)=C_1$,$u(1)=-\int_0^1\dfrac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}X(s)u(s)\mathrm{d}_qs+C_1+C_2$.因为 $D_qu(t)=-D_qI_q^{\alpha}(X(t)u(t))+C_2=-I_q^{\alpha-1}(X(t)u(t))+C_2=-\int_0^t\dfrac{(t-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}X(s)u(s)\mathrm{d}_qs+C_2$,故有 $D_qu(0)=C_2$,$D_qu(1)=-\int_0^1\dfrac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}X(s)u(s)\mathrm{d}_qs+C_2$.由边界条件 $u(0)-u'(0)=u(1)+u'(1)=0$,可得

$$\begin{cases}C_1=C_2,\\ -\int_0^1\dfrac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}X(s)u(s)\mathrm{d}_qs+C_1+C_2-\int_0^1\dfrac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}X(s)u(s)\mathrm{d}_qs+C_2=0,\end{cases}$$

故 $C_1=C_2=\dfrac{1}{3}\int_0^1\left[\dfrac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}+\dfrac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\right]X(s)u(s)\mathrm{d}_qs$.因此有

$$u(t)=-\int_0^t\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}X(s)u(s)\mathrm{d}_qs+\frac{1+t}{3}\int_0^1\left[\frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}+\frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\right]X(s)u(s)\mathrm{d}_qs=$$
$$\int_0^1\left[\frac{1+t}{3}\left(\frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}+\frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\right)-\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\right]X(s)u(s)\mathrm{d}_qs-\int_t^1\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}X(s)u(s)\mathrm{d}_qs.$$

当 $s<t$ 时,$\int_t^1\dfrac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}X(s)u(s)\mathrm{d}_qs=0$;当 $s\geqslant t$ 时,$-\int_0^1\dfrac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}X(s)u(s)\mathrm{d}_qs=-\left[\int_0^t\dfrac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}X(s)u(s)\mathrm{d}_qs+\int_t^1\dfrac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}X(s)u(s)\mathrm{d}_qs\right]=-\int_t^1\dfrac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}X(s)u(s)\mathrm{d}_qs$,即

$$\int_0^1\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}X(s)u(s)\mathrm{d}_qs=\int_t^1\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}X(s)u(s)\mathrm{d}_qs.$$

因此

$$u(t)=\int_0^1\left[\frac{1+t}{3}\left(\frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}+\frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\right)-\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\right]X(s)u(s)\mathrm{d}_qs=$$

$$\int_0^1 \frac{(1-qs)^{(\alpha-2)}}{3\Gamma_q(\alpha)}\left[(1+t)(1-sq^{\alpha-1}+[\alpha-1]_q)-\frac{3(t-qs)^{(\alpha-1)}}{(1-qs)^{(\alpha-2)}}\right]X(s)u(s)\mathrm{d}_qs=$$

$$\int_0^1 \frac{(1-qs)^{(\alpha-2)}(1-sq^{\alpha-1}+[\alpha-1]_q)}{3\Gamma_q(\alpha)}\left[(1+t)-\frac{3(t-qs)^{(\alpha-1)}}{(1-sq^{\alpha-1}+[\alpha-1]_q)(1-qs)^{(\alpha-2)}}\right]X(s)u(s)\mathrm{d}_qs=$$

$$\int_0^1 \frac{(1-qs)^{(\alpha-2)}(1-sq^{\alpha-1}+[\alpha-1]_q)}{3\Gamma_q(\alpha)}\left[(1+t)-\frac{3(t-qs)^{(\alpha-1)}}{(1-qs)^{(\alpha-1)}+[\alpha-1]_q(1-qs)^{(\alpha-2)}}\right]X(s)u(s)\mathrm{d}_qs.$$

引理 5 得证.

引理 6　对于任意的$(t,s)\in[0,1]\times(0,1)$,下式成立:

$$|H(t,s)|\leqslant \max\left\{2,\frac{q-q^{\alpha-1}}{q^{\alpha-1}-1}+1\right\}. \tag{4}$$

证明　1) 显然,当 $0\leqslant t\leqslant s\leqslant 1$ 时,有 $0\leqslant h_2(t,s)\leqslant 2$.

2) 当 $0\leqslant s\leqslant t\leqslant 1$ 时,因为 $h_1(t,s)=1+t-\dfrac{3(t-qs)^{(\alpha-1)}}{(1-qs)^{(\alpha-1)}+[\alpha-1]_q(1-qs)^{(\alpha-2)}}$,所以由性质 1 可得${}_tD_qh_1(t,s)=1-\dfrac{3[\alpha-1]_q(t-qs)^{(\alpha-2)}}{(1-qs)^{(\alpha-1)}+[\alpha-1]_q(1-qs)^{(\alpha-2)}}$.

下面证明 $[{}_tD_qh_1(t,s)]_{t\to s^+}<0$, 即:

$$(1-qs)^{(\alpha-1)}+[\alpha-1]_q(1-qs)^{(\alpha-2)}<3[\alpha-1]_q(s-qs)^{(\alpha-2)},$$

$$(1-qs)^{(\alpha-1)}<3[\alpha-1]_qs^{\alpha-2}(1-q)^{(\alpha-2)}-[\alpha-1]_q(1-qs)^{(\alpha-2)}. \tag{5}$$

因为$(1-qs)^{(\alpha-1)}$ 关于 s 递减,故有$(1-q)^{(\alpha-1)}\leqslant(1-qs)^{(\alpha-1)}\leqslant 1$. 又因为$(1-qs)^{(\alpha-2)}$ 关于 s 递增,故有 $1\leqslant(1-qs)^{(\alpha-2)}\leqslant(1-q)^{(\alpha-2)}$. 若使式(5) 成立,只要满足下式即可:

$$1<3[\alpha-1]_qs^{\alpha-2}(1-q)^{(\alpha-2)}-[\alpha-1]_q(1-q)^{(\alpha-2)}.$$

又因为 $s^{\alpha-2}$ 关于 s 递增,故有 $s^{\alpha-2}\geqslant 1$. 因此,若使式(5) 成立,需满足下式:

$$1<3[\alpha-1]_q(1-q)^{(\alpha-2)}-[\alpha-1]_q(1-q)^{(\alpha-2)}=2[\alpha-1]_q(1-q)^{(\alpha-2)}=$$

$$2\,\frac{1-q^{\alpha-1}}{1-q}\prod_{i=0}^{\infty}\frac{1-q^{i+1}}{1-q^{i+\alpha-1}}=2\prod_{i=0}^{\infty}\frac{1-q^{i+2}}{1-q^{i+\alpha}}.$$

由于 $\alpha\in(1,2]$, $q\in(0,1)$,因此 $q^{i+2}<q^{i+\alpha}$, $\dfrac{1-q^{i+2}}{1-q^{i+\alpha}}>1$,即式(5) 成立. 由此可知 $[{}_tD_qh_1(t,s)]_{t\to s^+}\in(-\infty,0]$.

固定区间(0,1) 上的点 s. 当 $t\in[s,1]$ 时,有

$${}_tD_qh_1(t,s)|_{t=1}=1-\frac{3[\alpha-1]_q(1-qs)^{(\alpha-2)}}{(1-qs)^{(\alpha-1)}+[\alpha-1]_q(1-qs)^{(\alpha-2)}}=1-\frac{3[\alpha-1]_q}{1-q^{\alpha-1}s+[\alpha-1]_q}.$$

令 $a^*=(1-2[\alpha-1]_q)q^{1-\alpha}$,以下分两种情况讨论.

(Ⅰ) 如果 $a^*\leqslant 0$, 则 $1-2[\alpha-1]_q\leqslant 0\Leftrightarrow 1+q-2q^{\alpha-1}\geqslant 0$ 且

$${}_tD_qh_1(t,s)|_{t=1}=1-\frac{3[\alpha-1]_q}{1-q^{\alpha-1}s+[\alpha-1]_q}=\frac{(1-2[\alpha-1]_q)-q^{\alpha-1}s}{1-q^{\alpha-1}s+[\alpha-1]_q}\leqslant$$

$$\frac{-q^{\alpha-1}s}{1-q^{\alpha-1}s+[\alpha-1]_q}\leqslant 0,\ s\in(0,1).$$

因为$-(t-qs)^{(\alpha-2)}$ 关于 t 是增函数,所以${}_tD_qh_1(t,s)=1-\dfrac{3[\alpha-1]_q(t-qs)^{(\alpha-2)}}{(1-qs)^{(\alpha-1)}+[\alpha-1]_q(1-qs)^{(\alpha-2)}}\leqslant 1-\dfrac{3[\alpha-1]_q}{1-q^{\alpha-1}s+[\alpha-1]_q}={}_tD_qh_1(t,s)|_{t=1}\leqslant 0$, $s\leqslant t$. 因此$h_1(1,s)=2-\dfrac{3(1-qs)^{(\alpha-1)}}{(1-qs)^{(\alpha-1)}+[\alpha-1]_q(1-qs)^{(\alpha-2)}}\leqslant h_1(t,s)\leqslant h_1(s,s)\leqslant 2$. 由 $1-2[\alpha-1]_q\leqslant 0$ 可得$h_1(1,s)=2-\dfrac{3(1-qs)^{(\alpha-1)}}{(1-qs)^{(\alpha-1)}+[\alpha-1]_q(1-qs)^{(\alpha-2)}}=\dfrac{(2[\alpha-1]_q-1)+q^{\alpha-1}s}{1-q^{\alpha-1}s+[\alpha-1]_q}\geqslant\dfrac{q^{\alpha-1}s}{1-q^{\alpha-1}s+[\alpha-1]_q}\geqslant 0$. 因此 $0\leqslant h_1(t,s)\leqslant 2$.

(Ⅱ) 如果 $0<a^*\leqslant 1$,则有以下两种情况:

i) 若 $a^* \leqslant s < 1$，有 $1-2[\alpha-1]_q \leqslant q^{\alpha-1}s$，且

$$ {}_tD_q h_1(t,s)\big|_{t=1} = 1 - \frac{3[\alpha-1]_q}{1-q^{\alpha-1}s+[\alpha-1]_q} = \frac{(1-2[\alpha-1]_q)-q^{\alpha-1}s}{1-q^{\alpha-1}s+[\alpha-1]_q} \leqslant 0. $$

同理，可以得出 $h_1(1,s) \leqslant h_1(t,s) \leqslant h_1(s,s) \leqslant 2$. 由 $1-2[\alpha-1]_q \leqslant q^{\alpha-1}s$ 得

$$ 2(1-q^{\alpha-1}s+[\alpha-1]_q)-3(1-q^{\alpha-1}s)=q^{\alpha-1}s-(1-2[\alpha-1]_q) \geqslant q^{\alpha-1}s-q^{\alpha-1}s=0, $$

即 $h_1(1,s) \geqslant 0$. 因此 $0 \leqslant h_1(t,s) \leqslant 2$.

ii) 若 $0 < s < a^*$，有 $1-2[\alpha-1]_q \geqslant q^{\alpha-1}s$，且

$$ {}_tD_q h_1(t,s)\big|_{t=1} = 1 - \frac{3[\alpha-1]_q}{1-q^{\alpha-1}s+[\alpha-1]_q} = \frac{(1-2[\alpha-1]_q)-q^{\alpha-1}s}{1-q^{\alpha-1}s+[\alpha-1]_q} \geqslant 0. $$

因此，$\exists t^* \in (s,1)$，s. t. ${}_tD_q h_1(t,s)\big|_{t=t^*}=0$，$h_1(s,s) \geqslant 0$，$h_1(1,s)=\dfrac{(2[\alpha-1]_q-1)+q^{\alpha-1}s}{1-q^{\alpha-1}s+[\alpha-1]_q} \leqslant 0$，即 $h_1(t^*,s) \leqslant h_1(1,s) \leqslant h_1(s,s) \leqslant 2$. 因此

$$ |h_1(t^*,s)| \leqslant \max\{-h_1(t^*,s),2\}. \tag{6} $$

观察 ${}_tD_q h_1(t,s)\big|_{t=t^*}=0 \Leftrightarrow 3[\alpha-1]_q(t^*-qs)^{(\alpha-2)}=(1-qs)^{(\alpha-1)}+[\alpha-1]_q(1-qs)^{(\alpha-2)}$，可知

$$ h_1(t^*,s)=1+t^*-\frac{3(t^*-qs)^{(\alpha-1)}}{3[\alpha-1]_q(t^*-qs)^{(\alpha-2)}}=1+t^*-\frac{t^*-q^{\alpha-1}s}{[\alpha-1]_q}= $$

$$ \left(1-\frac{1}{[\alpha-1]_q}\right)t^*+\frac{q^{\alpha-1}s}{[\alpha-1]_q}+1 \geqslant \frac{[\alpha-1]_q-1}{[\alpha-1]_q}\cdot 1+\frac{q^{\alpha-1}\cdot 0}{[\alpha-1]_q}+1=\frac{q-q^{\alpha-1}}{1-q^{\alpha-1}}+1. $$

利用上述不等式和式(6)，有 $|h_1(t,s)| \leqslant \max\left\{2,\dfrac{q-q^{\alpha-1}}{q^{\alpha-1}-1}+1\right\}$. 引理 6 证毕.

定理 1 如果方程(＊)有一个非平凡的连续解，那么

$$ \int_0^1 (1-qs)^{(\alpha-2)}(1-q^{\alpha-1}s+[\alpha-1]_q)|X(s)|\,\mathrm{d}_q s \geqslant \frac{3\Gamma_q(\alpha)}{\max\left\{2,\dfrac{q-q^{\alpha-1}}{q^{\alpha-1}-1}+1\right\}}. $$

证明 令 $B=C[0,1]$ 为巴拿赫空间，并赋予范数 $\|x\|_\infty=\max\limits_{0\leqslant t\leqslant 1}|x(t)|$，$x \in B$. 根据引理 5，对于任意的 $t\in[0,1]$，有 $u(t)=\dfrac{1}{3\Gamma_q(\alpha)}\displaystyle\int_0^1(1-qs)^{(\alpha-2)}(1-q^{\alpha-1}s+[\alpha-1]_q)H(t,s)X(s)u(s)\,\mathrm{d}_q s$. 再由引理 6 有 $\|u\|_\infty \leqslant \|u\|_\infty \dfrac{\max\left\{2,\dfrac{q-q^{\alpha-1}}{q^{\alpha-1}-1}+1\right\}}{3\Gamma_q(\alpha)} \times \displaystyle\int_0^1(1-qs)^{(\alpha-2)}(1-q^{\alpha-1}s+[\alpha-1]_q)|X(s)|\,\mathrm{d}_q s$. 定理 1 证毕.

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