文章编号：1004-4353(2018)04-0292-10

一类带有扰动项的分数阶q-差分方程解的存在唯一性

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摘要： 研究一类带有扰动项的非线性分数阶q-差分方程边值问题. 首先给出了该问题解的表达式，并分析了格林函数的性质；然后利用混合单调算子不动点定理获得了该问题解的存在唯一性，并且构造了两个迭代序列的逼近解.

关键词： 分数阶q-差分方程；扰动项；混合单调算子

中图分类号： O175.6 **文献标识码：** A

Existence and uniqueness of solutions for a class of fractional q-differences equation with perturbation

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Abstract： In the article, we study a class of fractional q-differences equation boundary value problems with perturbation are discussed. Firstly, the expression of solutions is presented, and some characteristics of the Green function were analyzed. Secondly, by applying mixed monotone operators fixed point theorems, our results can not only guarantee the existence and uniqueness of solutions, but aslo construct two iterative sequences to approximate the solution.

Keywords： fractional q-differences equations; perturbation; mixed monotone operators

0 引言

1910 年，Jackson 引入了q-微积分概念[1]，之后 Al-Salam 给出了分数阶q-微积分的基本概念和基本性质[2]. 近年来，分数阶q-差分边值问题受到国内外学者的关注，并取得了一些研究成果[3-6]，但对于含有扰动项的分数阶q-差分边值问题研究得较少. 本文将探讨带有扰动项的q-差分边值问题：

$$\begin{cases} D_q^\alpha x(t) + f(t, x(t), D_q^\gamma x(t)) + g(t, x(t)) = 0; \\ D_q^i x(0) = 0, \ 0 \leqslant i \leqslant n-2; \\ D_q^\beta x(1) = k(x(1)). \end{cases} \tag{1}$$

其中 $t \in (0,1)$, $n-1 < \alpha \leqslant n$, $n > 3$ 且 $1 \leqslant \gamma \leqslant \beta \leqslant n-2$, $g: [0,1] \times [0, +\infty) \to [0, +\infty)$, $f: [0,1] \times [0, +\infty) \times [0, +\infty) \to [0, +\infty)$, $k: [0, +\infty) \to [0, +\infty)$ 为连续函数.

收稿日期： 2018－11－13
通信作者： 侯成敏(1963—)，女，教授，研究方向为微分方程理论及其应用.

1　预备知识

定义 1[2]　Riemann-Liouville 型 q-积分定义为$I_q^p h(t)=\dfrac{1}{\Gamma_q(p)}\int_0^t (t-qs)^{(p-1)}h(s)\mathrm{d}_q s$，$p>0$，$h\in C[0,1]$.

定义 2[2]　Riemann-Liouville 型 q-导数定义为$D_q^p h(t)=(D_q^m I_q^{m-p}h)(t)$，$p>0$，$h\in C[0,1]$，$m$ 是不小于 p 的整数.

引理 1[6]　假设 $h\in C[0,1]$，$\alpha\geqslant\beta\geqslant 0$，则$D_q^\beta I_q^\alpha h(t)=I_q^{\alpha-\beta}h(t)$.

引理 2[6]　令 $h\in C[0,1]\cap L^1[0,1]$，$\alpha>0$，则$I_q^\alpha D_q^\alpha h(t)=c_1 t^{\alpha-1}+c_2 t^{\alpha-2}+\cdots+c_n t^{\alpha-n}$，其中 $c_i\in\mathbf{R}$，$i=1,2,3,\cdots,n\ (n=[\alpha]+1)$.

以下给出一些符号和已知结果，具体内容参考文献[7-12]. 假设$(E,\|\cdot\|)$是实 Banach 空间，θ 是 E 的零元素. 非空的闭凸集 $P\subset E$ 称为锥，如果满足如下条件：(a) $x\in P$，$\lambda\geqslant 0\Rightarrow\lambda x\in P$；(b) $x\in P$，$-x\in P\Rightarrow x=\theta$. 则 E 就定义了序，$x\leqslant y$ 当且仅当 $y-x\in P$. 令 $\mathring{P}=\{x\in P\mid x$ 为 P 内点$\}$，若 $\mathring{P}$ 非空，则称 P 为实锥. 锥 P 称为正规的，如果存在一个常数 $N>0$，使得对于任意的 $\forall x,y\in E$，$\theta\leqslant x\leqslant y$ 使得$\|x\|\leqslant N\|y\|$；在此情况下，N 叫做 P 的正规常数. 称算子 $A:E\to E$ 是单增(单减)的，如果对于 $x\leqslant y$ 使得 $Ax\leqslant Ay\ (Ax\geqslant Ay)$. 对于任意 $x,y\in E$，$x\sim y$ 表示存在$\lambda>0$ 和 $\mu>0$，使得 $\lambda x\leqslant y\leqslant\mu x$，所以"$\sim$"是一个等价关系. 对于 $h>\theta\ (h\geqslant\theta, h\neq 0)$，定义 $P_h=\{x\in E\mid x\sim h\}$，容易得出 $P_h\subset P$.

定义 3[13]　$A:P\times P\to P$ 称为混合单调算子，如果$A(x,y)$固定 $y\in P$ 关于x 是单增的，固定$x\in P$ 关于 y 是单减的，即如果 $\forall x_i,y_i\in P\ (i=1,2)$，$x_1\leqslant x_2$，$y_1\geqslant y_2$，则有 $A(x_1,y_1)\leqslant A(x_2,y_2)$. 当 $A(x,x)=x$，称 x 是 A 的不动点.

定义 4　$A:P\to P$ 称为亚齐次算子，如果 $A(tx)\geqslant tAx$，$\forall t\in(0,1)$，$x\in P$.

定义 5　令 $D=P$ 或 $D=\mathring{P}$，存在实数 α 且 $0\leqslant\alpha<1$，$A:D\to D$ 称为α-凹算子，如果 $A(tx)\geqslant t^\alpha Ax$，$\forall t\in(0,1)$，$x\in D$.

引理 3[14]　令 $\alpha\in(0,1)$，$P\subset E$ 为正规锥，$A:P\to P$ 为增亚齐次算子，$B:P\to P$ 为单减算子，$C:P\times P\to P$ 为混合单调算子，且 B 和C 满足$B(t^{-1}y)\geqslant tBy$，$C(tx,t^{-1}y)\geqslant t^\alpha C(x,y)$，$\forall t\in(0,1)$，$x,y\in P$. 假设：

(A_1) 存在 $h_0\in P_h$ 使得 $A(h_0,h_0)\in P_h$，$Bh_0\in P$，$C(h_0,h_0)\in P_h$；

(A_2) 存在常数 $\delta>0$ 使得 $C(x,y)\geqslant\delta(Ax+By)$，$\forall x,y\in P$.

则有：

1) $A:P_h\to P_h$，$B:P_h\to P_h$，$C:P_h\times P_h\to P_h$；

2) 存在 $u_0,v_0\in P_h$，$r\in(0,1)$ 使得 $rv_0\leqslant u_0<v_0$，$u_0\leqslant Au_0+Bv_0+C(u_0,v_0)\leqslant Av_0+Bu_0+C(v_0,u_0)\leqslant v_0$；

3) 算子方程 $Ax+Bx+C(x,x)=x$ 有唯一解 $x^*\in P_h$；

4) 任意初值 $u_0,v_0\in P_h$，构造序列 $u_n=Au_{n-1}+Bv_{n-1}+C(u_{n-1},v_{n-1})$ 和 $v_n=Av_{n-1}+Bu_{n-1}+C(v_{n-1},u_{n-1})$，其中 $n=1,2,\cdots$，于是有 $u_n\to x^*$，$v_n\to x^*\ (n\to\infty)$.

引理 4[14]　令 $\alpha\in(0,1)$，$P\subset E$ 为正规锥，$A:P\to P$ 为增亚齐次算子，$B:P\to P$ 为单减算子，$C:P\times P\to P$ 为混合单调算子，且 B 和C 满足$B(t^{-1}y)\geqslant t^\alpha By$，$C(tx,t^{-1}y)\geqslant tC(x,y)$，$\forall t\in(0,1)$，$x,y\in P$. 假设：

(A_1') 存在 $h_0\in P_h$ 使得 $A(h_0,h_0)\in P_h$，$Bh_0\in P$，$C(h_0,h_0)\in P_h$；

(A_2') 存在常数 $\delta>0$ 使得 $Ax+C(x,y)\leqslant\delta By$，$\forall x,y\in P$.

则可得与引理 3 中 1)—4) 相同的结论.

引理 5[14] 令 $\alpha \in (0,1)$, $P \subset E$ 为正规锥, $A : P \to P$ 为 α-凹算子, $B : P \to P$ 为单减算子, $C : P \times P \to P$ 为混合单调算子, 且 B 和 C 满足 $B(t^{-1}y) \geqslant tBy$, $C(tx, t^{-1}y) \geqslant tC(x,y)$, $\forall t \in (0,1)$, $x, y \in P$. 假设:

(A_1'') 存在 $h_0 \in P_h$ 使得 $A(h_0, h_0) \in P_h$, $Bh_0 \in P$, $C(h_0, h_0) \in P_h$;

(A_2'') 存在常数 $\delta > 0$ 使得 $By + C(x,y) \leqslant \delta Ax$, $\forall x, y \in P$.

则可得与引理 3 中 1)—4) 相同的结论.

引理 6 令 $p(t) \in C[0,1]$, 方程

$$\begin{cases} D_q^{\alpha} x(t) + p(t) = 0, \ t \in (0,1), \ n-1 < \alpha \leqslant n; \\ D_q^{i} x(0) = 0, \ 0 \leqslant i \leqslant n-2; \\ D_q^{\beta} x(1) = k(x(1)), \ 1 \leqslant \beta \leqslant n-2 \end{cases} \tag{2}$$

有唯一解

$$x(t) = \int_0^1 G(t, qs) p(s) \mathrm{d}_q s + \frac{\Gamma_q(\alpha - \beta)}{\Gamma_q(\alpha)} k(x(1)) t^{\alpha-1}, \tag{3}$$

其中

$$G(t, qs) = \frac{1}{\Gamma_q(\alpha)} \begin{cases} t^{\alpha-1}(1-qs)^{(\alpha-\beta-1)} - (t-qs)^{(\alpha-1)}, \ 0 \leqslant qs \leqslant t \leqslant 1; \\ t^{\alpha-1}(1-qs)^{(\alpha-\beta-1)}, \ 0 \leqslant t \leqslant qs \leqslant 1 \end{cases} \tag{4}$$

为格林函数.

引理 7 格林函数(4) 具有以下一些性质:

$$0 \leqslant t^{\alpha-1}(1-qs)^{(\alpha-\beta-1)} [1 - (1 - sq^{\alpha-\beta})^{(\beta)}] \leqslant \Gamma_q(\alpha) G(t, qs) \leqslant t^{\alpha-1}(1-qs)^{(\alpha-\beta-1)}; \tag{5}$$

$$0 \leqslant t^{\alpha-\gamma-1}(1-qs)^{(\alpha-\beta-1)} [1 - (1 - sq^{\alpha-\beta})^{(\beta-\gamma)}] \leqslant \Gamma_q(\alpha-\gamma) D_q^{\gamma} G(t, qs) \leqslant t^{\alpha-\gamma-1}(1-qs)^{(\alpha-\beta-1)}. \tag{6}$$

证明 首先不等式(5) 的右边部分显然成立, 所以只需证明 $\Gamma_q(\alpha) G(t, qs)$ 的左边部分即可. 当 $0 \leqslant t \leqslant qs \leqslant 1$ 时, 其显然成立; 当 $0 \leqslant qs \leqslant t \leqslant 1$ 时,

$$\Gamma_q(\alpha) G(t, qs) = t^{\alpha-1}(1-qs)^{(\alpha-\beta-1)} - (t-qs)^{(\alpha-1)} \geqslant$$
$$t^{\alpha-1}(1-qs)^{(\alpha-\beta-1)} - t^{\alpha-1}(1-qs)^{(\alpha-1)} = t^{\alpha-1}(1-qs)^{(\alpha-\beta-1)} [1 - (1 - sq^{\alpha-\beta})^{(\beta)}] \geqslant 0,$$

其显然成立:因此, 不等式(5) 成立.

其次证明不等式(6) 成立. 由式(4) 知

$$D_q^{\gamma} G(t, qs) = \frac{1}{\Gamma_q(\alpha-\gamma)} \begin{cases} t^{\alpha-\gamma-1}(1-qs)^{(\alpha-\beta-1)} - (t-qs)^{(\alpha-\gamma-1)}, \ 0 \leqslant qs \leqslant t \leqslant 1; \\ t^{\alpha-\gamma-1}(1-qs)^{(\alpha-\beta-1)}, \ 0 \leqslant t \leqslant qs \leqslant 1, \end{cases} \tag{7}$$

显然不等式(6) 的右边部分成立, 所以只需证明 $\Gamma_q(\alpha-\gamma) D_q^{\gamma} G(t, qs)$ 的左边部分成立即可. 当 $0 \leqslant t \leqslant qs \leqslant 1$ 时, 其显然成立; 当 $0 \leqslant qs \leqslant t \leqslant 1$ 时,

$$\Gamma_q(\alpha-\gamma) D_q^{\gamma} G(t, qs) = t^{\alpha-\gamma-1}(1-qs)^{(\alpha-\beta-1)} - (t-qs)^{(\alpha-\gamma-1)} \geqslant$$
$$t^{\alpha-\gamma-1}(1-qs)^{(\alpha-\beta-1)} - t^{\alpha-\gamma-1}(1-qs)^{(\alpha-\gamma-1)} = t^{\alpha-\gamma-1}(1-qs)^{(\alpha-\beta-1)} [1 - (1 - sq^{\alpha-\beta})^{(\beta-\gamma)}] \geqslant 0,$$

其显然成立:因此, 不等式(6) 成立.

2 主要结果及其证明

令 $E = \{x \mid x \in C[0,1], D_q^{\gamma} x \in C[0,1]\}$ 为 Banach 空间, 且定义范数 $\|x\| = \max\{\max|x(t)|, \max D_q^{\gamma}|x(t)|, \forall t \in [0,1]\}$; 定义 E 中序关系 $u \preceq v$, 如果 $u(t) \leqslant v(t)$ 且 $D_q^{\gamma} u(t) \leqslant D_q^{\gamma} v(t)$; 令 $P = \{x \in E \mid x(t) \geqslant 0, D_q^{\gamma} x(t) \geqslant 0, \forall t \in [0,1]\}$, 则 P 是一个正规锥且 $P_h \subset E$.

方程(1) 有唯一解

$$x(t)=\int_0^1 G(t,qs)f(s,x(s),D_q^\gamma x(s))\mathrm{d}_q s+\int_0^1 G(t,qs)g(s,x(s))\mathrm{d}_q s+\frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha)}k(x(1))t^{\alpha-1},$$

其中 $G(t,qs)$ 由式(4) 给出. 定义如下算子：

$$A(u)(t)=\int_0^1 G(t,qs)g(s,u(s))\mathrm{d}_q s,\ t\in[0,1];\tag{8}$$

$$B(v)(t)=\frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha)}k(v(1))t^{\alpha-1},\ t\in[0,1];\tag{9}$$

$$C(u,v)(t)=\int_0^1 G(t,qs)f(s,u(s),D_q^\gamma v(s))\mathrm{d}_q s,\ t\in[0,1].\tag{10}$$

容易得到 $u=Au+Bu+C(u,u)$ 时，u 为方程(1) 的解.

定理 1　假设：

(H_1) $f:[0,1]\times[0,+\infty)\times[0,+\infty)\to[0,+\infty)$，$g:[0,1]\times[0,+\infty)\to[0,+\infty)$，且 $k:[0,+\infty)\to[0,+\infty)$ 为连续函数.

(H_2) $f(t,x,y)$ 固定 $t\in[0,1]$，$y\in[0,+\infty)$ 关于 $x\in[0,+\infty)$ 为单增的；固定 $t\in[0,1]$，$x\in[0,+\infty)$ 关于 $y\in[0,+\infty)$ 为单减的；$g(t,x)$ 固定 $t\in[0,1]$ 关于 $x\in[0,+\infty)$ 为单增的；$K(y)$ 固定 $t\in[0,1]$ 关于 $y\in[0,+\infty)$ 为单减的；且 $k(y(1))\neq 0$.

(H_3) 存在常数 $\alpha\in(0,1)$ 使得 $f(t,\lambda x,\lambda^{-1}y)\geqslant\lambda^\alpha f(t,x,y)$，$\forall\lambda\in(0,1)$，$t\in[0,1]$，$x,y\in[0,+\infty)$；$g(t,x)$ 和 $k(y)$ 满足不等式 $g(t,\lambda x)\geqslant\lambda g(t,x)$，$k(\lambda^{-1}y)\geqslant\lambda k(y)$，$\forall\lambda\in(0,1)$，$x,y\in[0,+\infty)$.

(H_4) $g(t,0)$ 不恒等于 0，$t\in[0,1]$，存在 $\delta_1>0$，$\delta_2>0$ 使得 $f(t,x,y)\geqslant\delta_1 g(t,x)$，$f(t,x,y)\geqslant\delta_2\geqslant k(y)$，$t\in[0,1]$，$x,y\in[0,+\infty)$.

则有：

1) 存在 $u_0,v_0\in P_h\subset E$，$r\in(0,1)$ 使得 $rv_0\leq u_0\prec v_0$，即 $rv_0\leqslant u_0<v_0$，$rD_q^\gamma v_0\leqslant D_q^\gamma u_0<D_q^\gamma v_0$：

$$u_0\leqslant\int_0^1 G(t,qs)f(s,u_0(s),D_q^\gamma v_0(s))\mathrm{d}_q s+\int_0^1 G(t,qs)g(s,u_0(s))\mathrm{d}_q s+\frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha)}k(v_0(1))t^{\alpha-1};$$

$$D_q^\gamma u_0\leqslant\int_0^1 D_q^\gamma G(t,qs)f(s,u_0(s),D_q^\gamma v_0(s))\mathrm{d}_q s+\int_0^1 D_q^\gamma G(t,qs)g(s,u_0(s))\mathrm{d}_q s+$$
$$\frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha-\gamma)}k(v_0(1))t^{\alpha-\gamma-1};$$

$$v_0\geqslant\int_0^1 G(t,qs)f(s,v_0(s),D_q^\gamma u_0(s))\mathrm{d}_q s+\int_0^1 G(t,qs)g(s,v_0(s))\mathrm{d}_q s+\frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha)}k(u_0(1))t^{\alpha-1};$$

$$D_q^\gamma v_0\geqslant\int_0^1 D_q^\gamma G(t,qs)f(s,v_0(s),D_q^\gamma u_0(s))\mathrm{d}_q s+\int_0^1 D_q^\gamma G(t,qs)g(s,v_0(s))\mathrm{d}_q s+$$
$$\frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha-\gamma)}k(u_0(1))t^{\alpha-\gamma-1};$$

其中 $h(t)=t^{\alpha-1}$，$\forall t\in[0,1]$，$G(t,qs)$ 由式(4) 给出.

2) 方程(1) 有唯一解 $x^*\in P_h$.

3) 任意初值 $u_0,v_0\in P_h$，构造序列：

$$u_n(t)=\int_0^1 G(t,qs)f(s,u_{n-1}(s),D_q^\gamma v_{n-1}(s))\mathrm{d}_q s+\int_0^1 G(t,qs)g(s,u_{n-1}(s))\mathrm{d}_q s+$$
$$\frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha)}k(v_{n-1}(1))t^{\alpha-1},\ n=1,2,\cdots;$$

$$v_n(t)=\int_0^1 G(t,qs)f(s,v_{n-1}(s),D_q^\gamma u_{n-1}(s))\mathrm{d}_q s+\int_0^1 G(t,qs)g(s,v_{n-1}(s))\mathrm{d}_q s+$$
$$\frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha)}k(u_{n-1}(1))t^{\alpha-1},\ n=1,2,\cdots.$$

进而有 $u_n \to x^*$, $v_n \to x^*(n \to \infty)$.

证明 为了得到结论,需要证明算子 A,B,C 满足引理 3 的全部条件.

首先证明 $A: P \to P$, $B: P \to P$ 和 $C: P \times P \to P$. 由(H_1)与引理 7 有 $\forall u,v \in P$, $Au(t) \geqslant 0$, $D_q^\gamma Au(t) \geqslant 0$, $Bv(t) \geqslant 0$, $D_q^\gamma Bv(t) \geqslant 0$, $C(u,v)(t) \geqslant 0$, $D_q^\gamma C(u,v)(t) \geqslant 0, \forall t \in [0,1]$, 因此 $Au \in P, Bv \in P$, $C(u,v) \in P$. 其次证明 A 为单增的, B 为单减的. $\forall u,v \in P$, $u \preceq v$ 得 $u(t) \leqslant v(t)$, $D_q^\gamma u(t) \leqslant D_q^\gamma v(t), \forall t \in [0,1]$. 由($H_2$)和 $G(t,qs) > 0$ 可得

$$A(u) = \int_0^1 G(t,qs)g(s,u(s))\mathrm{d}_q s \leqslant \int_0^1 G(t,qs)g(s,v(s))\mathrm{d}_q s = A(v);$$

$$D_q^\gamma A(u) = \int_0^1 D_q^\gamma G(t,qs)g(s,u(s))\mathrm{d}_q s \leqslant \int_0^1 D_q^\gamma G(t,qs)g(s,v(s))\mathrm{d}_q s = D_q^\gamma A(v);$$

$$B(u) = \frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha)}k(u(1))t^{\alpha-1} \geqslant \frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha)}k(v(1))t^{\alpha-1} = B(v);$$

$$D_q^\gamma B(u) = \frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha-\gamma)}k(u(1))t^{\alpha-\gamma-1} \geqslant \frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha-\gamma)}k(v(1))t^{\alpha-\gamma-1} = D_q^\gamma B(v);$$

因此 $Au \preceq Av$, $Bu \succ Bv$. 下证 C 为混合单调算子. $\forall u_1,u_2,v_1,v_2 \in P$, $u_1 \preceq u_2$, $v_1 \succeq v_2$, 即 $u_1(t) \leqslant u_2(t)$, $D_q^\gamma u_1(t) \leqslant D_q^\gamma u_2(t)$, $v_1(t) \geqslant v_2(t)$, $D_q^\gamma v_1(t) \geqslant D_q^\gamma v_2(t), \forall t \in [0,1]$. 由($H_2$)和 $G(t,qs) > 0$, 可得:

$$C(u_1,v_1)(t) = \int_0^1 G(t,qs)f(s,u_1(s),D_q^\gamma v_1(s))\mathrm{d}_q s \leqslant \int_0^1 G(t,qs)f(s,u_2(s),D_q^\gamma v_2(s))\mathrm{d}_q s = C(u_2,v_2);$$

$$D_q^\gamma C(u_1,v_1)(t) = \int_0^1 D_q^\gamma G(t,qs)f(s,u_1(s),D_q^\gamma v_1(s))\mathrm{d}_q s \leqslant \int_0^1 D_q^\gamma G(t,qs)f(s,u_2(s),D_q^\gamma v_2(s))\mathrm{d}_q s = D_q^\gamma C(u_2,v_2).$$

因此 $C(u_1,v_1) \preceq C(u_2,v_2)$. 以下证明 A 为亚齐次算子,且 B 和 C 满足 $B(t^{-1}y) \geqslant tBy$, $C(tx,t^{-1}y) \geqslant t^\alpha C(x,y)$. 根据($H_3$)和 $\lambda \in (0,1)$, 有:

$$A(\lambda u) = \int_0^1 G(t,qs)g(s,\lambda u(s))\mathrm{d}_q s \geqslant \lambda \int_0^1 G(t,qs)g(s,u(s))\mathrm{d}_q s = \lambda A(u);$$

$$D_q^\gamma A(\lambda u) = \int_0^1 D_q^\gamma G(t,qs)g(s,\lambda u(s))\mathrm{d}_q s \geqslant \lambda \int_0^1 D_q^\gamma G(t,qs)g(s,u(s))\mathrm{d}_q s = \lambda D_q^\gamma A(u).$$

因此 $A(\lambda u) \succeq \lambda Au$, $u \in P$, 则 A 为亚齐次算子. 同理可得下列不等关系成立:

$$B(\lambda^{-1}v) = \frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha)}k(\lambda^{-1}v(1))t^{\alpha-1} \geqslant \lambda \frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha)}k(v(1))t^{\alpha-1} = \lambda B(v);$$

$$D_q^\gamma B(\lambda^{-1}v) = \frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha-\gamma)}k(\lambda^{-1}v(1))t^{\alpha-\gamma-1} \geqslant \lambda \frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha-\gamma)}k(v(1))t^{\alpha-\gamma-1} = \lambda D_q^\gamma B(v);$$

$$C(\lambda u,\lambda^{-1}v)(t) = \int_0^1 G(t,qs)f(s,\lambda u(s),\lambda^{-1} D_q^\gamma v(s))\mathrm{d}_q s \geqslant \lambda^\alpha \int_0^1 G(t,qs)f(s,u(s),D_q^\gamma v(s))\mathrm{d}_q s = \lambda^\alpha C(u,v);$$

$$D_q^\gamma C(\lambda u,\lambda^{-1}v)(t) = \int_0^1 D_q^\gamma G(t,qs)f(s,\lambda u(s),\lambda^{-1} D_q^\gamma v(s))\mathrm{d}_q s \geqslant \lambda^\alpha \int_0^1 D_q^\gamma G(t,qs)f(s,u(s),D_q^\gamma v(s))\mathrm{d}_q s = \lambda^\alpha D_q^\gamma C(u,v).$$

因此 $B(\lambda^{-1}v) \succeq \lambda Bv$, $C(\lambda u,\lambda^{-1}v) \succeq \lambda^\alpha C(u,v)$.

再证明算子 A,B,C 满足引理 3 中的条件(A_1)和(A_2). 由(H_1)、(H_2)与式(5),可得:

$$A(h) = \int_0^1 G(t,qs)g(s,s^{\alpha-1})\mathrm{d}_q s \leqslant t^{\alpha-1}\int_0^1 \frac{(1-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha)}g(s,1)\mathrm{d}_q s;$$

$$A(h) \geqslant \int_0^1 \frac{t^{\alpha-1}(1-qs)^{(\alpha-\beta-1)}[1-(1-sq^{\alpha-\beta})^{(\beta)}]}{\Gamma_q(\alpha)} g(s,s^{\alpha-1})\mathrm{d}_q s \geqslant$$

$$t^{\alpha-1}\int_0^1 \frac{(1-qs)^{(\alpha-\beta-1)}[1-(1-sq^{\alpha-\beta})^{(\beta-\gamma)}]}{\Gamma_q(\alpha)} g(s,0)\mathrm{d}_q s;$$

$$C(h,h)(t)=\int_0^1 G(t,qs)f(s,s^{\alpha-1},D_q^{\gamma}(s^{\alpha-1}))\mathrm{d}_q s \leqslant$$

$$\int_0^1 \frac{t^{\alpha-1}}{\Gamma_q(\alpha)}(1-qs)^{(\alpha-\beta-1)} f\left(s,s^{\alpha-1},\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\gamma)}s^{\alpha-\gamma-1}\right)\mathrm{d}_q s \leqslant t^{\alpha-1}\int_0^1 \frac{(1-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha)} f(s,1,0)\mathrm{d}_q s;$$

$$C(h,h) \geqslant \int_0^1 \frac{t^{\alpha-1}(1-qs)^{(\alpha-\beta-1)}[1-(1-sq^{\alpha-\beta})^{(\beta)}]}{\Gamma_q(\alpha)} f\left(s,s^{\alpha-1},\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\gamma)}s^{\alpha-\gamma-1}\right)\mathrm{d}_q s \geqslant$$

$$t^{\alpha-1}\int_0^1 \frac{(1-qs)^{(\alpha-\beta-1)}[1-(1-sq^{\alpha-\beta})^{(\beta-\gamma)}]}{\Gamma_q(\alpha)} f\left(s,0,\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\gamma)}\right)\mathrm{d}_q s.$$

另一方面，

$$D_q^{\lambda}A(h)=\int_0^1 D_q^{\lambda}G(t,qs)g(s,s^{\alpha-1})\mathrm{d}_q s \leqslant \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\gamma)}t^{\alpha-\gamma-1}\int_0^1 \frac{(1-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha)} g(s,1)\mathrm{d}_q s;$$

$$D_q^{\gamma}A(h) \geqslant \int_0^1 \frac{t^{\alpha-\gamma-1}(1-qs)^{(\alpha-\beta-1)}[1-(1-sq^{\alpha-\beta})^{(\beta-\gamma)}]}{\Gamma_q(\alpha-\gamma)} g(s,s^{\alpha-1})\mathrm{d}_q s \geqslant$$

$$\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\gamma)}t^{\alpha-\gamma-1}\int_0^1 \frac{(1-qs)^{(\alpha-\beta-1)}[1-(1-sq^{\alpha-\beta})^{(\beta-\gamma)}]}{\Gamma_q(\alpha)} g(s,0)\mathrm{d}_q s;$$

$$D_q^{\gamma}C(h,h)(t)=\int_0^1 D_q^{\gamma}G(t,qs)f(s,s^{\alpha-1},D_q^{\gamma}(s^{\alpha-1}))\mathrm{d}_q s \leqslant$$

$$\int_0^1 \frac{t^{\alpha-\gamma-1}}{\Gamma_q(\alpha-\gamma)}(1-qs)^{(\alpha-\beta-1)} f\left(s,s^{\alpha-1},\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\gamma)}s^{\alpha-\gamma-1}\right)\mathrm{d}_q s \leqslant$$

$$\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\gamma)}t^{\alpha-\gamma-1}\int_0^1 \frac{(1-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha)} f(s,1,0)\mathrm{d}_q s;$$

$$D_q^{\gamma}C(h,h) \geqslant \int_0^1 \frac{t^{\alpha-\gamma-1}(1-qs)^{(\alpha-\beta-1)}[1-(1-sq^{\alpha-\beta})^{(\beta-\gamma)}]}{\Gamma_q(\alpha-\gamma)} f\left(s,s^{\alpha-1},\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\gamma)}s^{\alpha-\gamma-1}\right)\mathrm{d}_q s \geqslant$$

$$\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\gamma)}t^{\alpha-\gamma-1}\int_0^1 \frac{(1-qs)^{(\alpha-\beta-1)}[1-(1-sq^{\alpha-\beta})^{(\beta-\gamma)}]}{\Gamma_q(\alpha)} f\left(s,0,\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\gamma)}\right)\mathrm{d}_q s.$$

令

$$c_1=\int_0^1 \frac{(1-qs)^{(\alpha-\beta-1)}[1-(1-sq^{\alpha-\beta})^{(\beta-\gamma)}]}{\Gamma_q(\alpha)} g(s,0)\mathrm{d}_q s;$$

$$c_2=\int_0^1 \frac{(1-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha)} g(s,1)\mathrm{d}_q s; \tag{11}$$

$$c_3=\int_0^1 \frac{(1-qs)^{(\alpha-\beta-1)}[1-(1-sq^{\alpha-\beta})^{(\beta-\gamma)}]}{\Gamma_q(\alpha)} f\left(s,0,\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\gamma)}\right)\mathrm{d}_q s;$$

$$c_4=\int_0^1 \frac{(1-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha)} f(s,1,0)\mathrm{d}_q s. \tag{12}$$

由(H_2)和(H_4)可得 $c_2 \geqslant c_1 > 0$，$c_4 \geqslant c_3 \geqslant \delta_1 c_1 > 0$，因此 $c_1 h \leq Ah \leq c_2 h$，$c_3 h \leq C(h,h) \leq c_4 h$，所以 $Ah \in P_h$，$C(h,h) \in P_h$. 另外，有：

$$B(h)=\frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha)}k(h(1))t^{\alpha-1}=\frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha)}k(1)h(t); \tag{13}$$

$$D_q^{\gamma}B(h)=\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\gamma)}t^{\alpha-\gamma-1}\,\frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha)}k(1)=\frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha)}k(1)\,D_q^{\gamma}h(t). \tag{14}$$

因为 $k(y(1)) \neq 0$，所以可得 $Bh \in P_h$. 满足引理 3 中的条件(A_1).

下证算子 A,B,C 满足引理 3 中的条件(A_2). 由(H_4)可得：

$$C(u,v)(t)=\int_0^1 G(t,qs)f(s,u(s),D_q^\gamma v(s))\mathrm{d}_q s\geqslant\delta_1\int_0^1 G(t,qs)g(s,u(s))\mathrm{d}_q s=\delta_1 Au;$$

$$D_q^\gamma C(u,v)(t)=\int_0^1 D_q^\gamma G(t,qs)f(s,u(s),D_q^\gamma v(s))\mathrm{d}_q s\geqslant\delta_1\int_0^1 D_q^\gamma G(t,qs)g(s,u(s))\mathrm{d}_q s=\delta_1 D_q^\gamma Au.$$

因此 $C(u,v)\geq\delta_1 Au$. 再根据(H_4)和引理 7 有:

$$C(u,v)(t)=\int_0^1 G(t,qs)f(s,u(s),D_q^\gamma v(s))\mathrm{d}_q s\geqslant\frac{\delta_2 t^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^1[(1-qs)^{(\alpha-\beta-1)}-(1-qs)^{(\alpha-1)}]\mathrm{d}_q s\geqslant$$

$$\frac{t^{\alpha-1}}{\Gamma_q(\alpha)}k(v(1))\int_0^1[(1-qs)^{(\alpha-\beta-1)}-(1-qs)^{(\alpha-\gamma-1)}]\mathrm{d}_q s=$$

$$\frac{1}{\Gamma_q(\alpha-\beta)}\int_0^1[(1-qs)^{(\alpha-\beta-1)}-(1-qs)^{(\alpha-\gamma-1)}]\mathrm{d}_q s\,Bv;$$

$$D_q^\gamma C(u,v)(t)=\int_0^1 D_q^\gamma G(t,qs)f(s,u(s),D_q^\gamma v(s))\mathrm{d}_q s\geqslant$$

$$\frac{\delta_2 t^{\alpha-\gamma-1}}{\Gamma_q(\alpha-\gamma)}\int_0^1[(1-qs)^{(\alpha-\beta-1)}-(1-qs)^{(\alpha-\gamma-1)}]\mathrm{d}_q s\geqslant\frac{t^{\alpha-\gamma-1}}{\Gamma_q(\alpha-\gamma)}k(v(1))\cdot$$

$$\int_0^1[(1-qs)^{(\alpha-\beta-1)}-(1-qs)^{(\alpha-\gamma-1)}]\mathrm{d}_q s=\frac{1}{\Gamma_q(\alpha-\beta)}\int_0^1[(1-qs)^{(\alpha-\beta-1)}-(1-qs)^{(\alpha-\gamma-1)}]\mathrm{d}_q s\,D_q^\gamma Bv.$$

因此

$$C(u,v)\geq\frac{1}{\Gamma_q(\alpha-\beta)}\int_0^1[(1-qs)^{(\alpha-\beta-1)}-(1-qs)^{(\alpha-\gamma-1)}]\mathrm{d}_q s\,Bv.$$

取 $\delta=\frac{1}{2}\min\left\{\delta_1,\frac{1}{\Gamma_q(\alpha-\beta)}\int_0^1[(1-qs)^{(\alpha-\beta-1)}-(1-qs)^{(\alpha-\gamma-1)}]\mathrm{d}_q s\right\}$,则 $C(u,v)\geq\delta(Au+Bv)$. 由此知算子 A,B,C 满足引理 3 的全部条件.

由引理 3 中的结论 2) 可以得到存在 $u_0,v_0\in P_h\subset E$, $r\in(0,1)$ 使得 $rv_0\preceq u_0\prec v_0$,即 $rv_0\leqslant u_0<v_0$, $rD_q^\gamma v_0\leqslant D_q^\gamma u_0<D_q^\gamma v_0$:

$$u_0\leqslant\int_0^1 G(t,qs)f(s,u_0(s),D_q^\gamma v_0(s))\mathrm{d}_q s+\int_0^1 G(t,qs)g(s,u_0(s))\mathrm{d}_q s+\frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha)}k(v_0(1))t^{\alpha-1};$$

$$D_q^\gamma u_0\leqslant\int_0^1 D_q^\gamma G(t,qs)f(s,u_0(s),D_q^\gamma v_0(s))\mathrm{d}_q s+\int_0^1 D_q^\gamma G(t,qs)g(s,u_0(s))\mathrm{d}_q s+$$

$$\frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha-\gamma)}k(v_0(1))t^{\alpha-\gamma-1};$$

$$v_0\geqslant\int_0^1 G(t,qs)f(s,v_0(s),D_q^\gamma u_0(s))\mathrm{d}_q s+\int_0^1 G(t,qs)g(s,v_0(s))\mathrm{d}_q s+\frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha)}k(u_0(1))t^{\alpha-1};$$

$$D_q^\gamma v_0\geqslant\int_0^1 D_q^\gamma G(t,qs)f(s,v_0(s),D_q^\gamma u_0(s))\mathrm{d}_q s+\int_0^1 D_q^\gamma G(t,qs)g(s,v_0(s))\mathrm{d}_q s+$$

$$\frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha-\gamma)}k(u_0(1))t^{\alpha-\gamma-1}.$$

由引理 3 中的结论 3) 知方程(1)有唯一解 $x^*\in P_h$;由引理 3 中的结论 4),对任意初值 $u_0,v_0\in P_h$,构造 2 个迭代序列 $\{u_n\}$ 和 $\{v_n\}$,当 $n\to\infty$ 时 $u_n\to x^*$, $v_n\to x^*$:

$$u_n(t)=\int_0^1 G(t,qs)f(s,u_{n-1}(s),D_q^\gamma v_{n-1}(s))\mathrm{d}_q s+\int_0^1 G(t,qs)g(s,u_{n-1}(s))\mathrm{d}_q s+$$

$$\frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha)}k(v_{n-1}(1))t^{\alpha-1},\ n=1,2,\cdots;$$

$$v_n(t)=\int_0^1 G(t,qs)f(s,v_{n-1}(s),D_q^\gamma u_{n-1}(s))\mathrm{d}_q s+\int_0^1 G(t,qs)g(s,v_{n-1}(s))\mathrm{d}_q s+$$

$$\frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha)}k(u_{n-1}(1))t^{\alpha-1},\ n=1,2,\cdots.$$

定理 2　假设满足定理 1 中的(H_1) 和(H_2)，且：

(H_3') 存在常数 $\alpha\in(0,1)$，使得 $k(\lambda^{-1}y)\geqslant\lambda^{\alpha}k(y)$，$\forall\lambda\in(0,1)$，$y\in[0,+\infty)$；$f(t,x,y)$，$g(t,x)$ 满足不等式 $f(t,\lambda x,\lambda^{-1}y)\geqslant\lambda f(t,x,y)$，$g(t,\lambda x)\geqslant\lambda g(t,x)$，$\forall\lambda\in(0,1)$，$x,y\in[0,+\infty)$，$t\in[0,1]$；

(H_4') $g(t,0)$ 不恒等于 0，$f\left(t,0,\dfrac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\beta)}\right)$ 不恒等于 0，$t\in[0,1]$，存在常数 $\delta_0>0$，使得 $g(t,x)+f(t,x,y)\leqslant\delta_0\leqslant k(y)$，$t\in[0,1]$，$x,y\in[0,+\infty)$.

则可得与定理 1 中 1)—3) 相同的结论.

证明　与定理 1 证明相似，由(H_1) 和(H_2) 得 $A:P\to P$，$B:P\to P$ 和 $C:P\times P\to P$ 且 A 为单增的，B 为单减的，C 为混合单调算子. 由(H_3') 得：

$$A(\lambda u)=\int_0^1 G(t,qs)g(s,\lambda u(s))\mathrm{d}_q s\geqslant\lambda\int_0^1 G(t,qs)g(s,u(s))\mathrm{d}_q s=\lambda A(u);$$

$$D_q^{\gamma}A(\lambda u)=\int_0^1 D_q^{\gamma}G(t,qs)g(s,\lambda u(s))\mathrm{d}_q s\geqslant\lambda\int_0^1 D_q^{\gamma}G(t,qs)g(s,u(s))\mathrm{d}_q s=\lambda D_q^{\gamma}A(u);$$

$$B(\lambda^{-1}v)=\frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha)}k(\lambda^{-1}v(1))t^{\alpha-1}\geqslant\lambda^{\alpha}\ \frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha)}k(v(1))t^{\alpha-1}=\lambda^{\alpha}B(v);$$

$$D_q^{\gamma}B(\lambda^{-1}v)=\frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha-\gamma)}k(\lambda^{-1}v(1))t^{\alpha-\gamma-1}\geqslant\lambda^{\alpha}\ \frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha-\gamma)}k(v(1))t^{\alpha-\gamma-1}=\lambda^{\alpha}D_q^{\gamma}B(v);$$

$$C(\lambda u,\lambda^{-1}v)(t)=\int_0^1 G(t,qs)f(s,\lambda u(s),\lambda^{-1}\ D_q^{\gamma}v(s))\mathrm{d}_q s\geqslant\lambda\int_0^1 G(t,qs)f(s,u(s),D_q^{\gamma}v(s))\mathrm{d}_q s=\lambda C(u,v);$$

$$D_q^{\gamma}C(\lambda u,\lambda^{-1}v)(t)=\int_0^1 D_q^{\gamma}G(t,qs)f(s,\lambda u(s),\lambda^{-1}\ D_q^{\gamma}v(s))\mathrm{d}_q s\geqslant\lambda\int_0^1 D_q^{\gamma}G(t,qs)f(s,u(s),D_q^{\gamma}v(s))\mathrm{d}_q s=\lambda D_q^{\gamma}C(u,v).$$

因此 $A(\lambda u)\geq\lambda Au$，$B(\lambda^{-1}v)\geq\lambda^{\alpha}Bv$，$C(\lambda u,\lambda^{-1}v)\geq\lambda C(u,v)$，由此知 A 为亚齐次算子. 由于 $f\left(t,0,\dfrac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\beta)}\right)$ 不恒等于 0 与(H_2) 有 $c_4\geqslant c_3>0$，显然 $C(h,h)\in P_h$；又由 $g(t,0)$ 不恒等于 0 与(H_2) 有 $c_2\geqslant c_1>0$，则 $Ah\in P_h$，其中 c_1,c_2,c_3,c_4 如式(11)、(12)；由式(13)、(14) 和 $k(y(1))\neq0$，可得 $Bh\in P_h$.

下证算子 A,B,C 满足引理 4 中的(A_2') 条件. 由(H_4') 和引理 7，对于 $t\in[0,1]$，$x,y\in[0,+\infty)$，可得：

$$Au+C(u,v)=\int_0^1 G(t,qs)(g(s,u(s))+f(s,u(s),D_q^{\gamma}v(s)))\mathrm{d}_q s\leqslant\int_0^1\frac{t^{\alpha-1}(1-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha)}\delta_0\mathrm{d}_q s\leqslant\frac{t^{\alpha-1}}{\Gamma_q(\alpha)}k(v(1))\int_0^1(1-qs)^{(\alpha-\beta-1)}\mathrm{d}_q s=\frac{1}{\Gamma_q(\alpha-\beta)}\int_0^1(1-qs)^{(\alpha-\beta-1)}\mathrm{d}_q s Bv;$$

$$D_q^{\gamma}(Au+C(u,v))=\int_0^1 D_q^{\gamma}G(t,qs)(g(s,u(s))+f(s,u(s),D_q^{\gamma}v(s)))\mathrm{d}_q s\leqslant\int_0^1\frac{t^{\alpha-\gamma-1}(1-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\gamma)}\delta_0\mathrm{d}_q s\leqslant\frac{t^{\alpha-\gamma-1}}{\Gamma_q(\alpha-\gamma)}k(v(1))\int_0^1(1-qs)^{(\alpha-\beta-1)}\mathrm{d}_q s=\frac{1}{\Gamma_q(\alpha-\beta)}\int_0^1(1-qs)^{(\alpha-\beta-1)}\mathrm{d}_q s D_q^{\gamma}Bv.$$

令 $\delta=\dfrac{1}{\Gamma_q(\alpha-\beta)}\displaystyle\int_0^1(1-qs)^{(\alpha-\beta-1)}\mathrm{d}_q s$，则有 $Au+C(u,v)\leq\delta Bv$. 由引理 4，可得与定理 1 中 1)—3) 相同的结论.

定理 3　假设满足定理 1 中的(H_1) 和(H_2)，且：

(H_3'') 存在常数 $\alpha\in(0,1)$ 使得 $g(t,\lambda x)\geqslant\lambda^{\alpha}g(t,x)$, $\forall\lambda\in(0,1)$, $t\in[0,1]$, $x\in[0,+\infty)$; $f(t,x,y)$ 和 $k(y)$ 满足不等式 $f(t,\lambda x,\lambda^{-1}y)\geqslant\lambda f(t,x,y)$, $k(\lambda^{-1}y)\geqslant\lambda k(y)$, $\forall\lambda\in(0,1)$, $x,y\in[0,+\infty)$, $t\in[0,1]$.

(H_4'') $f\left(t,0,\dfrac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\beta)}\right)$ 不恒等于 0, $t\in[0,1]$, 存在常数 δ_1 和 δ_2, 使得 $k(y)\leqslant\delta_1\leqslant g(t,x)$, $f(t,x,y)\leqslant\delta_2 g(t,x)$, $t\in[0,1]$, $x,y\in[0,+\infty)$.

则可得与定理 1 中 1)—3) 相同的结论.

证明 与定理 1 证明相似, 由 (H_1) 和 (H_2) 得 $A:P\to P$ 为单增的, $B:P\to P$ 为单减的, 且 $C:P\times P\to P$ 为混合单调算子. 下证算子 A,B,C 满足引理 5 的全部条件. 由 (H_3'') 得:

$$A(\lambda u)=\int_0^1 G(t,qs)g(s,\lambda u(s))\mathrm{d}_q s\geqslant\lambda^{\alpha}\int_0^1 G(t,qs)g(s,u(s))\mathrm{d}_q s=\lambda^{\alpha}A(u);$$

$$D_q^{\gamma}A(\lambda u)=\int_0^1 D_q^{\gamma}G(t,qs)g(s,\lambda u(s))\mathrm{d}_q s\geqslant\lambda^{\alpha}\int_0^1 D_q^{\gamma}G(t,qs)g(s,u(s))\mathrm{d}_q s=\lambda^{\alpha}D_q^{\gamma}A(u);$$

$$B(\lambda^{-1}v)=\frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha)}k(\lambda^{-1}v(1))t^{\alpha-1}\geqslant\lambda\frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha)}k(v(1))t^{\alpha-1}=\lambda B(v);$$

$$D_q^{\gamma}B(\lambda^{-1}v)=\frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha-\gamma)}k(\lambda^{-1}v(1))t^{\alpha-\gamma-1}\geqslant\lambda\frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha-\gamma)}k(v(1))t^{\alpha-\gamma-1}=\lambda D_q^{\gamma}B(v);$$

$$C(\lambda u,\lambda^{-1}v)(t)=\int_0^1 G(t,qs)f(s,\lambda u(s),\lambda^{-1}D_q^{\gamma}v(s))\mathrm{d}_q s\geqslant\lambda\int_0^1 G(t,qs)f(s,u(s),D_q^{\gamma}v(s))\mathrm{d}_q s=\lambda C(u,v);$$

$$D_q^{\gamma}C(\lambda u,\lambda^{-1}v)(t)=\int_0^1 D_q^{\gamma}G(t,qs)f(s,\lambda u(s),\lambda^{-1}D_q^{\gamma}v(s))\mathrm{d}_q s\geqslant\lambda\int_0^1 D_q^{\gamma}G(t,qs)f(s,u(s),D_q^{\gamma}v(s))\mathrm{d}_q s=\lambda D_q^{\gamma}C(u,v).$$

因此 $A(\lambda u)\geq\lambda^{\alpha}Au$, $B(\lambda^{-1}v)\geq\lambda Bv$, $C(\lambda u,\lambda^{-1}v)\geq\lambda C(u,v)$. 由此知 A 为 α-凹算子. 由 (H_2) 有 $c_4\geqslant c_3>0$ 和 $c_1\geqslant c_3>0$, 其中 c_1,c_2,c_3,c_4 如式(11)、(12), 所以 $C(h,h)\in P_h$, $Ah\in P_h$. 又由定理 1 证明得 $Bh\in P_h$, 因此满足引理 5 中的条件 (A_1'').

下证算子 A,B,C 满足引理 5 中的 (A_2'') 条件. 由 (H_4'') 与引理 7, 对于 $t\in[0,1]$, $x,y\in[0,+\infty)$, 可得:

$$C(u,v)(t)=\int_0^1 G(t,qs)f(s,u(s),D_q^{\gamma}v(s))\mathrm{d}_q s\leqslant\delta_2\int_0^1 G(t,qs)g(s,u(s))\mathrm{d}_q s=\delta_2 Au;$$

$$D_q^{\gamma}C(u,v)(t)=\int_0^1 D_q^{\gamma}G(t,qs)f(s,u(s),D_q^{\gamma}v(s))\mathrm{d}_q s\leqslant\delta_2\int_0^1 D_q^{\gamma}G(t,qs)g(s,u(s))\mathrm{d}_q s=\delta_2 D_q^{\gamma}Au.$$

因此, $C(u,v)\leq\delta_2 Au$. 另一方面, 有:

$$A(u)=\int_0^1 G(t,qs)g(s,u(s))\mathrm{d}_q s\geqslant\int_0^1\frac{t^{\alpha-1}(1-qs)^{(\alpha-\beta-1)}[1-(1-sq^{\alpha-\beta})^{(\beta)}]}{\Gamma_q(\alpha)}\delta_1\mathrm{d}_q s\geqslant\frac{t^{\alpha-1}}{\Gamma_q(\alpha)}k(v(1))\int_0^1[(1-qs)^{(\alpha-\beta-1)}-(1-qs)^{(\alpha-\gamma-1)}]\mathrm{d}_q s=\frac{1}{\Gamma_q(\alpha-\beta)}\int_0^1[(1-qs)^{(\alpha-\beta-1)}-(1-qs)^{(\alpha-\gamma-1)}]\mathrm{d}_q s\,Bv;$$

$$D_q^{\gamma}A(u)=\int_0^1 D_q^{\gamma}G(t,qs)g(s,u(s))\mathrm{d}_q s\geqslant\int_0^1\frac{t^{\alpha-\gamma-1}(1-qs)^{(\alpha-\beta-1)}[1-(1-sq^{\alpha-\beta})^{(\beta-\gamma)}]}{\Gamma_q(\alpha-\gamma)}\delta_1\mathrm{d}_q s\geqslant\frac{t^{\alpha-\gamma-1}}{\Gamma_q(\alpha-\gamma)}k(v(1))\int_0^1[(1-qs)^{(\alpha-\beta-1)}-(1-qs)^{(\alpha-\gamma-1)}]\mathrm{d}_q s=\frac{1}{\Gamma_q(\alpha-\beta)}\int_0^1[(1-qs)^{(\alpha-\beta-1)}-(1-qs)^{(\alpha-\gamma-1)}]\mathrm{d}_q s\,D_q^{\gamma}Bv.$$

令 $\delta'=\dfrac{\Gamma_q(\alpha-\beta)}{\int_0^1[(1-qs)^{(\alpha-\beta-1)}-(1-qs)^{(\alpha-\gamma-1)}]\mathrm{d}_qs}$,则 $Bv\leq\delta'Au$.取 $\delta=2\max\{\delta',\delta_2\}$,则 $Bv+C(u,v)\leq\delta Au$.应用引理 5 可得与定理 1 中 1)—3) 相同的结论.

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