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# 一类非线性时滞分数阶 $q$ -差分系统的有限时间稳定性

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**摘要:** 研究了一类非线性时滞分数阶  $q$ -差分系统的有限时间稳定性. 首先, 证明了一类非线性时滞 Caputo 型分数阶  $q$ -差分系统解的存在唯一性; 其次, 利用  $q$ -Mittag-Leffler 函数和 Gronwall 不等式建立了该时滞分数阶  $q$ -差分系统有限时间稳定性的充分条件. 最后通过举例表明了本文主要结果的有效性和实用性.  
**关键词:** 时滞分数阶  $q$ -差分;  $q$ -Mittag-Leffler 函数; Gronwall 不等式; 有限时间稳定性  
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## Finite time stability of a class of nonlinear delay fractional $q$ -difference systems

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**Abstract:** In this paper, finite time stability of a class of nonlinear delay fractional  $q$ -difference systems is studied. Firstly, we prove the existence and uniqueness for the solutions of a class of nonlinear delay Caputo fractional  $q$ -difference systems. Secondly, by using  $q$ -Mittag-Leffler function and the Gronwall inequality, sufficient conditions for the finite time stability of the delay fractional  $q$ -difference system is obtained. Finally, an example is presented to illustrate the validity and practicability of our main results.  
**Keywords:** delay fractional  $q$ -difference;  $q$ -Mittag-Leffler function; the Gronwall inequality; finite time stability

近年来,随着分数阶 $q$ -微积分理论的发展,时标上的分数阶 $q$ -微积分理论引起了广大研究者的兴趣<sup>[1-11]</sup>. 2016 年, Thabet Abdeljawad 等<sup>[11]</sup>利用  $q$ -Mittag-Leffler 函数证明了一个新的关于分数阶 $q$ -微积分的 Gronwall 不等式,并利用这个不等式证明了形如

$$\begin{cases} {}_qC_a^\alpha x(t) = A_0x(t) + A_1x(\tau t) + f(t, x(t), x(\tau t)), & t \in [a, \infty)_q, \\ x(t) = \varphi(t), & t \in I_\tau \end{cases}$$

的时标上的分数阶 $q$ -差分系统解的存在唯一性,同时给出了这个解的估计不等式,其中 $0 < \alpha < 1$ . 本文在文献[11]的启发下,研究形如

$$\begin{cases} {}_qC_a^\alpha (x(t) - \lambda x(\tau t)) = \mathbf{A}x(t) + \mathbf{B}x(\tau t) + f(t, x(t), x(\tau t)), & t \in [a, \infty)_q, \\ x(t) = \varphi(t), & t \in I_\tau, \\ \nabla_q x(t) = \psi(t), & t \in I_\tau \end{cases} \tag{1}$$

的时标上的分数阶 $q$ -差分系统解的唯一性及其有限时间稳定性. 其中: $0 < q < 1$ ;  $1 < \alpha < 2$ ;  $\lambda$  为参数,且 $0 < \lambda < 1$ ;  ${}_qC_a^\alpha$  表示 Caputo 左  $q$ -分数阶导数. 记时标  $T_q = \{q^n : n \in \mathbf{Z}\} \cup \{0\}$ ,  $a = q^{n_0}$ ,  $n_0 \in \mathbf{Z}$ ,

$T_a = [a, \infty)_q = \{q^{-i}a : i=0, 1, 2, \dots\}$ ,  $T_{\tau a} = [\tau a, \infty)_q = \{\tau a, q^{-1}\tau a, q^{-2}\tau a, q^{-3}\tau a, \dots\}$ , 其中  $\tau = q^d \in T_q$ ,  $d \in \mathbf{N}_0$ ,  $\mathbf{N}_0 = \{0, 1, 2, \dots\}$ ,  $I_\tau = \{\tau a, q^{-1}\tau a, q^{-2}\tau a, q^{-3}\tau a, \dots, a\}$ .  $\mathbf{A}, \mathbf{B} \in \mathbf{R}^{n \times n}$  为常数矩阵, 且  $\|\mathbf{A}\| = \max_{1 \leq j \leq n} \sum_{i=1}^n |a_{ij}|$  是矩阵的范数.  $x(t)$  是  $n$  元状态向量,  $x : T_{\tau a} \rightarrow \mathbf{R}^n$ ,  $\|x\| = \sup_{t \in T_{\tau a}} \|x(t)\|$ . 初始函数  $\varphi, \psi : I_\tau \rightarrow \mathbf{R}^n$ ,  $\|\varphi\| = \sup_{t \in I_\tau} \|\varphi(t)\|$ ,  $\|\psi\| = \sup_{t \in I_\tau} \|\psi(t)\|$ . 非线性函数  $f : T_{\tau a} \times \mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ , 设  $D(T_{\tau a} \times \mathbf{R}^n \times \mathbf{R}^n, \mathbf{R}^n)$  表示  $T_{\tau a} \times \mathbf{R}^n \times \mathbf{R}^n$  到  $\mathbf{R}^n$  上的有界函数构成的集合, 记为  $D$ , 显然  $D$  为 Banach 空间.

## 1 预备知识

**定义 1**<sup>[11]</sup> 对于函数  $f : T_q \rightarrow \mathbf{R}$ , 定义函数  $f$  的 nabla  $q$ -导数为  $\nabla_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}$ ,  $t \in T_q - \{0\}$ .

定义函数  $f$  的 nabla  $q$ -积分为  $\int_0^t f(s) \nabla_q s = (1-q)t \sum_{n=0}^{\infty} f(tq^n)q^n$ , 并且对于  $0 \leq a \in T_q$  有

$$\int_a^t f(s) \nabla_q s = \int_0^t f(s) \nabla_q s - \int_0^a f(s) \nabla_q s.$$

**定义 2**<sup>[11]</sup> 对于  $n \in \mathbf{N}$ , 定义  $q$ -分数阶函数为  $(t-s)_q^n = \prod_{k=0}^{n-1} (t-sq^k)$ ,  $n \in \mathbf{N}$ ; 对于非正整数  $\alpha$ , 定义  $q$ -分数阶函数为  $(t-s)_q^\alpha = t^\alpha \prod_{n=0}^{\infty} \frac{t-sq^n}{t-sq^{\alpha+n}}$ .

**定义 3**<sup>[11]</sup> 对于函数  $f : T_q \rightarrow \mathbf{R}$ , 定义函数  $f$  的左  $q$ -分数阶积分为:

$${}_q \nabla_a^{-\alpha} f(t) = \frac{1}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{\alpha-1} f(s) \nabla_q s,$$

其中  $\alpha \neq 0, -1, -2, \dots$ , 积分起点为  $a$ , 且  $0 < a \in T_q$ . 这里  $q$ -伽玛函数满足  $\Gamma_q(\alpha+1) = \frac{1-q^\alpha}{1-q} \Gamma_q(\alpha)$ ,  $\Gamma_q(1) = 1$ ,  $\alpha > 0$ .

**定义 4**<sup>[11]</sup> 设  $\alpha > 0$ ,  $\alpha \notin \mathbf{N}$ , 对于函数  $f : T_q \rightarrow \mathbf{R}$ , 定义函数  $f$  的  $\alpha$  阶 Caputo 左  $q$ -分数阶导数为:

$${}_q C_a^\alpha f(t) = {}_q \nabla_a^{-(n-\alpha)} \nabla_q^n f(t) = \frac{1}{\Gamma_q(n-\alpha)} \int_a^t (t-qs)_q^{n-\alpha-1} \nabla_q^n f(s) \nabla_q s,$$

其中  $n = [\alpha] + 1$ ,  $[\alpha]$  表示不超过  $\alpha$  的最大整数. 当  $\alpha = n \in \mathbf{N}$  时, 记  ${}_q C_a^\alpha f(t) = \nabla_q^n f(t)$ . 定义左 Riemann 型  $q$ -分数阶导数为  $({}_q \nabla_a^\alpha f)(t) = (\nabla_q^n {}_q \nabla_a^{-(n-\alpha)} f)(t)$ .

**定义 5**<sup>[11]</sup> 对于  $z, z_0 \in C$ , 且  $\Re(\alpha) > 0$ , 定义  $q$ -Mittag-Leffler 函数为:

$${}_q E_{\alpha, \beta}(\lambda, z - z_0) = \sum_{k=0}^{\infty} \lambda^k \frac{(z - z_0)_q^{ak}}{\Gamma_q(ak + \beta)},$$

当  $\beta = 1$  时, 记  ${}_q E_a(\lambda, z - z_0) = {}_q E_{a, 1}(\lambda, z - z_0)$ .

**引理 1**<sup>[11]</sup> 对于  $\alpha, \beta, \gamma \in \mathbf{R}$ , 有:

$$(i) (t-s)_q^{\beta+\gamma} = (t-s)_q^\beta (t-s)_q^\gamma;$$

$$(ii) (at-as)_q^\beta = a^\beta (t-s)_q^\beta;$$

$$(iii) q\text{-分数阶函数关于 } t \text{ 的 nabla } q\text{-导数为 } \nabla_q (t-s)_q^\alpha = \frac{1-q^\alpha}{1-q} (t-s)_q^{\alpha-1};$$

$$(iv) {}_q \nabla_a^{-\alpha} (x-a)_q^\mu = \frac{\Gamma_q(\mu+1)}{\Gamma_q(\alpha+\mu+1)} (x-a)_q^{\mu+\alpha}, 0 < \alpha < x.$$

**引理 2**<sup>[11]</sup> 设  $\alpha > 0$ , 有  ${}_q \nabla_a^{-\alpha} {}_q C_a^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)_q^k}{\Gamma_q(k+1)} \nabla_q^k f(a)$ ; 若  $0 < \alpha \leq 1$ , 则有

$${}_q \nabla_a^{-\alpha} {}_q C_a^\alpha f(t) = f(t) - f(a).$$

**引理 3**<sup>[11]</sup> (Gronwall 不等式) 设  $\alpha > 0$ , 对于  $t \in T_a = [a, \infty)_q$ ,  $a = q^{n_0}$ ,  $n_0$  为某一整数,  $u(t), v(t)$ ,

$w(t)$  是非负函数,其中  $v(t)$  和  $w(t)$  是非递减函数,且存在常数  $M$  使得  $w(t) \leq M$ . 如果  $u(t) \leq v(t) + w(t) {}_q\nabla_a^{-\alpha} u(t)$ , 那么  $u(t) \leq v(t) {}_qE_a(w(t)\Gamma_q(\alpha), t-a)$ ,  $t \in T_a$ .

**引理 4**<sup>[12]</sup> 对于分数阶 $q$ -差分系统(1),若初始函数  $\|\varphi\| < \delta$  时,有  $\|x(t)\| < \epsilon$ ,  $\forall t \in J = [a, b]_q$ , 则称系统(1)是关于  $\{a, J, \delta, \epsilon, \tau\}$  有限时间稳定的. 其中  $\delta$  是正实数,  $\epsilon \in \mathbf{R}$ ,  $\delta < \epsilon$ ,  $J = [a, b]_q \subset T_a$ ,  $b = aq^{-i_0}$ ,  $i_0$  为某一正整数.

为了方便后面的证明,本文分别用  $(H_1)$  和  $(H_2)$  表示如下两个条件:

$(H_1)$   $f \in D(T_{\tau a} \times \mathbf{R}^n \times \mathbf{R}^n, \mathbf{R}^n)$  是一个 Lipschitz 型函数,即存在一个正常数  $L_1$ , 使得

$$\|f(t, x(t), x(\tau t)) - f(t, y(t), y(\tau t))\| \leq L_1(\|x(t) - y(t)\| + \|x(\tau t) - y(\tau t)\|), \quad t \in T_a.$$

$(H_2)$  存在一个正常数  $L_2$ , 使得  $\|f(t, x(t), x(\tau t))\| \leq L_2$ .

## 2 解的唯一性

**定理 1**  $x: T_{\tau a} \rightarrow \mathbf{R}^n$  是系统(1)的解当且仅当

$$x(t) = \varphi(a) - \lambda \varphi(\tau a) + (\psi(a) - \lambda \tau \psi(\tau a))(t-a) + \lambda x(\tau t) + \frac{1}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{\alpha-1} [Ax(s) + Bx(\tau s) + f(s, x(s), x(\tau s))] \nabla_q s, \quad t \in T_a, \quad (2)$$

且  $x(t) = \varphi(t)$ ,  $\nabla_q x(t) = \psi(t)$ ,  $t \in I_\tau$ .

**证明** 对于  $t \in I_\tau$ , 显然系统(1)的解  $x(t) = \varphi(t)$ ; 对于  $t \in T_a$ , 利用引理 2 有

$$x(t) - \lambda x(\tau t) = x(a) - \lambda x(\tau a) + (\nabla_q x(t) - \lambda \nabla_q x(\tau t)) \Big|_{t=a} (t-a) + {}_q\nabla_a^{-\alpha} [Ax(t) + Bx(\tau t) + f(t, x(t), x(\tau t))],$$

即  $x(t) = \varphi(a) - \lambda \varphi(\tau a) + (\psi(a) - \lambda \tau \psi(\tau a))(t-a) + \lambda x(\tau t) + \frac{1}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{\alpha-1} [Ax(s) + Bx(\tau s) + f(s, x(s), x(\tau s))] \nabla_q s$ ,  $t \in T_a$ .

**定理 2** 假设条件  $(H_1)$  成立, 如果  $x(t)$  和  $y(t)$  是系统(1)的两个解, 那么  $x(t) = y(t)$ .

**证明** 设  $x(t)$  和  $y(t)$  是系统(1)的两个解, 记  $z(t) = x(t) - y(t)$ . 显然当  $t \in I_\tau$  时有  $z(t) = 0$ . 对于  $t \in T_a$ , 有  $z(t) = \lambda z(\tau t) + \frac{1}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{\alpha-1} [Az(s) + Bz(\tau s) + f(s, x(s), x(\tau s)) - f(s, y(s), y(\tau s))] \nabla_q s$ . 其中, 当  $t \in I_\tau^{-1} = \{a, q^{-1}a, q^{-2}a, q^{-3}a, \dots, \tau^{-1}a\}$  时, 有  $z(\tau t) = 0$ . 此时

$$z(t) = \frac{1}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{\alpha-1} [Az(s) + f(s, x(s), x(\tau s)) - f(s, y(s), y(\tau s))] \nabla_q s,$$

因此有

$$\begin{aligned} \|z(t)\| &\leq \frac{1}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{\alpha-1} [\|A\| \|z(s)\| + \|f(s, x(s), x(\tau s)) - f(s, y(s), y(\tau s))\|] \nabla_q s \leq \\ &\frac{1}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{\alpha-1} [\|A\| \|z(s)\| + L_1(\|x(t) - y(t)\| + \|x(\tau t) - y(\tau t)\|)] \nabla_q s = \\ &\frac{1}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{\alpha-1} (\|A\| + L_1) \|z(t)\| \nabla_q s \leq \frac{\|A\| + L_1}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{\alpha-1} \|z(t)\| \nabla_q s. \end{aligned}$$

由引理 3 有  $\|z(t)\| \leq 0 \cdot {}_qE_a((\|A\| + L_1)\Gamma_q(\alpha), t-a) = 0$ , 即  $x(t) = y(t)$ ,  $t \in I_\tau^{-1}$ .

当  $t \in [\tau^{-1}a, \infty)_q$  时, 有  $z(t) = \lambda z(\tau t) + \frac{1}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{\alpha-1} [Az(s) + f(s, x(s), x(\tau s)) - f(s, y(s), y(\tau s))] \nabla_q s + \frac{1}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{\alpha-1} Bz(\tau s) \nabla_q s$ , 因此, 有

$$\|z(t)\| \leq \lambda \|z(\tau t)\| + \frac{\|A\| + L_1}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{\alpha-1} \|z(t)\| \nabla_q s + \frac{\|B\| + L_1}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{\alpha-1} \|z(\tau s)\| \nabla_q s. \quad (3)$$

记  $I = \{\tau, q^{-1}\tau, q^{-2}\tau, q^{-3}\tau, \dots, 1\}$ , 设  $\bar{z}(t) = \sup_{\theta \in I} \|x(\theta t)\|$ , 则由式(3)有

$$\bar{z}(t) \leq \frac{\|A\| + \|B\| + 2L_1}{\Gamma_q(\alpha)(1-\lambda)} \int_a^t (t - qs)_q^{\alpha-1} \bar{z}(t) \nabla_q s.$$

由引理 3 有  $\|z(t)\| \leq \bar{z}(t) \leq 0$ .  ${}_q E_a \left( \frac{\|A\| + \|B\| + 2L_1}{1-\lambda} \Gamma_q(\alpha), t-a \right) = 0$ , 即  $x(t) = y(t)$ ,  $t \in [\tau^{-1}a, \infty)_q$ .

综上所述, 有  $x(t) = y(t)$ ,  $t \in T_a$ . 定理证毕.

### 3 有限时间稳定性

**定理 3** 假设条件  $(H_2)$  成立, 如果对  $\forall t \in J$  如下式 (4) 成立, 则系统 (1) 是关于  $\{a, J, \delta, \epsilon, \tau\}$  有限时间稳定的:

$$\left[ \frac{1 + \lambda + M_1(1 + \lambda\tau)(b-a)}{1-\lambda} + \frac{\|A\| + \|B\| + M_2}{(1-\lambda)\Gamma_q(\alpha+1)} (t-a)_q^\alpha \right] {}_q E_a \left( \frac{\|A\| + \|B\|}{1-\lambda} \Gamma_q(\alpha), t-a \right) \leq \frac{\epsilon}{\delta}, \quad (4)$$

其中  $M_1 \geq \frac{\|\psi\|}{\|\varphi\|}$ ,  $M_2 \geq \frac{L_2}{\|\varphi\|}$ ,  $J = [a, b]_q \subset T_a$ ,  $b = aq^{-i_0}$ ,  $i_0$  为某一正整数.

**证明** 对于  $t \in J$ , 系统 (1) 的解为

$$x(t) = \varphi(a) - \lambda\varphi(\tau a) + (\psi(a) - \lambda\tau\psi(\tau a))(t-a) + \lambda x(\tau t) + \frac{1}{\Gamma_q(\alpha)} \int_a^t (t - qs)_q^{\alpha-1} [Ax(s) + Bx(\tau s) + f(s, x(s), x(\tau s))] \nabla_q s, \quad t \in T_a,$$

且  $x(t) = \varphi(t)$ ,  $\nabla_q x(t) = \psi(t)$ ,  $t \in I_\tau$ . 于是有

$$\|x(t)\| \leq (1 + \lambda)\|\varphi\| + (1 + \lambda\tau)\|\psi\|(b-a) + \lambda \sup_{\theta \in I} \|x(\theta t)\| + \frac{\|A\| + \|B\|}{\Gamma_q(\alpha)} \int_a^t (t - qs)_q^{\alpha-1} (\sup_{\theta \in I} \|x(\theta s)\| + \|\varphi\|) \nabla_q s + \frac{L_2}{\Gamma_q(\alpha)} \int_a^t (t - qs)_q^{\alpha-1} \nabla_q s,$$

由于  $M_1 \geq \frac{\|\psi\|}{\|\varphi\|}$ ,  $M_2 \geq \frac{L_2}{\|\varphi\|}$ , 因此有

$$\begin{aligned} \|x(t)\| &\leq (1 + \lambda)\|\varphi\| + (1 + \lambda\tau)M_1\|\varphi\|(b-a) + \lambda \sup_{\theta \in I} \|x(\theta t)\| + \\ &\frac{\|A\| + \|B\|}{\Gamma_q(\alpha)} \int_a^t (t - qs)_q^{\alpha-1} \sup_{\theta \in I} \|x(\theta s)\| \nabla_q s + \frac{(\|A\| + \|B\| + M_2)\|\varphi\|}{\Gamma_q(\alpha+1)} (t-a)_q^\alpha = \\ &\|\varphi\| \left[ 1 + \lambda + (1 + \lambda\tau)M_1(b-a) + \frac{\|A\| + \|B\| + M_2}{\Gamma_q(\alpha+1)} (t-a)_q^\alpha \right] + \\ &\lambda \sup_{\theta \in I} \|x(\theta t)\| + \frac{\|A\| + \|B\|}{\Gamma_q(\alpha)} \int_a^t (t - qs)_q^{\alpha-1} \sup_{\theta \in I} \|x(\theta s)\| \nabla_q s. \end{aligned}$$

进一步有  $\|x(t)\| \leq \sup_{\theta \in I} \|x(\theta t)\| \leq \frac{\|\varphi\|}{1-\lambda} \left[ 1 + \lambda + (1 + \lambda\tau)M_1(b-a) + \frac{\|A\| + \|B\| + M_2}{\Gamma_q(\alpha+1)} (t-a)_q^\alpha \right] +$

$\frac{\|A\| + \|B\|}{(1-\lambda)\Gamma_q(\alpha)} \int_a^t (t - qs)_q^{\alpha-1} \sup_{\theta \in I} \|x(\theta s)\| \nabla_q s$ . 由引理 3 有

$$\begin{aligned} \|x(t)\| &\leq \sup_{\theta \in I} \|x(\theta t)\| \leq \frac{\|\varphi\|}{1-\lambda} \left[ 1 + \lambda + (1 + \lambda\tau)M_1(b-a) + \frac{\|A\| + \|B\| + M_2}{\Gamma_q(\alpha+1)} (t-a)_q^\alpha \right] \cdot \\ &{}_q E_a \left( \frac{\|A\| + \|B\|}{1-\lambda} \Gamma_q(\alpha), t-a \right), \end{aligned}$$

因此当  $\|\varphi\| < \delta$  时, 有  $\|x(t)\| < \epsilon$ ,  $\forall t \in J = [a, b]_q$ ; 所以, 系统 (1) 是关于  $\{a, J, \delta, \epsilon, \tau\}$  有限时间稳定的.

### 4 应用举例

**例 1** 考虑如下分数阶  $q$ -差分系统:

$$\begin{cases} {}_{1/2}C_{1/16}^{3/2}(x(t) - \lambda x(\tau t)) = 2x(t) + 3x(\tau t) - \cos x(t) + 3 \cos x(\tau t), & t \in [\frac{1}{16}, \infty)_q; \\ x(t) = \sin 2t, & t \in I_\tau; \\ \nabla_q x(t) = \frac{4}{t} \cos \frac{3t}{2} \sin \frac{t}{2}, & t \in I_\tau. \end{cases} \quad (5)$$

其中  $q = \frac{1}{2}$ ,  $\tau = \frac{1}{4}$ ,  $\alpha = \frac{3}{2}$ ,  $a = \frac{1}{16}$ ,  $b = \frac{1}{2}$ ,  $A = 2$ ,  $B = 3$ ,  $\varphi(t) = \sin 2t$ ,  $\psi(t) = \frac{4}{t} \cos \frac{3t}{2} \sin \frac{t}{2}$ ,  
 $f(t, x(t), x(\tau t)) = -\cos x(t) + 3 \cos x(\tau t)$ .

首先, 表明  $f(t, x(t), x(\tau t))$  满足条件  $(H_1)$ . 由于  $\|f(t, x(t), x(\tau t)) - f(t, y(t), y(\tau t))\| \leq \|-\cos x(t) + 3 \cos x(\tau t) + \cos y(t) - 3 \cos y(\tau t)\| \leq 3(\|\cos x(t) - \cos y(t)\| + \|\cos x(\tau t) - \cos y(\tau t)\|)$ , 因此, 条件  $(H_1)$  成立, 且  $L_1 = 3$ . 由定理 2 知系统 (5) 有唯一解. 其次, 表明  $f(t, x(t), x(\tau t))$  满足条件  $(H_2)$ . 由于  $\|f(t, x(t), x(\tau t))\| = \|-\cos x(t) + 3 \cos x(\tau t)\| \leq 4$ , 因此, 条件  $(H_2)$  成立, 且  $L_2 = 4$ . 因为  $\|\varphi\| = \sin \frac{1}{8}$ ,  $M_1 \geq \frac{\|\psi\|}{\|\varphi\|}$ ,  $M_2 \geq \frac{L_2}{\|\varphi\|}$ , 所以取  $M_1 = 256 \csc \frac{1}{8}$ ,  $M_2 = 4 \csc \frac{1}{8}$ ,  $\delta = 1$ . 于是对于某个参数  $\lambda$ ,  $0 < \lambda < 1$ , 有

$$\begin{aligned} & \left[ \frac{1 + \lambda + M_1(1 + \lambda\tau)(b - a)}{1 - \lambda} + \frac{\|A\| + \|B\| + M_2}{(1 - \lambda)\Gamma_q(\alpha + 1)}(t - a)_q^\alpha \right] {}_q E_\alpha \left( \frac{\|A\| + \|B\|}{1 - \lambda} \Gamma_q(\alpha), t - a \right) \leq \\ & \left[ \frac{1 + \lambda + 112(1 + \lambda\tau) \csc \frac{1}{8}}{1 - \lambda} + \frac{5 + 4 \csc \frac{1}{8}}{(1 - \lambda)\Gamma_q(5/2)}(t - a)_q^\alpha \right] {}_q E_\alpha \left( \frac{5}{1 - \lambda} \Gamma_q\left(\frac{3}{2}\right), t - a \right) \leq \\ & \left[ \frac{1 + \lambda + 112(1 + \lambda\tau) \csc \frac{1}{8}}{1 - \lambda} + \frac{5 + 4 \csc \frac{1}{8}}{(1 - \lambda)\Gamma_q(5/2)}(b - a)_q^\alpha \right] \sum_{k=0}^{\infty} \left( \frac{5}{1 - \lambda} \Gamma_q\left(\frac{3}{2}\right) \right)^k \frac{(b - a)_q^{\frac{3}{2}k}}{\Gamma_q\left(\frac{3}{2}k + 1\right)}. \end{aligned}$$

所以当  $\epsilon$  满足

$$\left[ \frac{1 + \lambda + 112(1 + \lambda\tau) \csc \frac{1}{8}}{1 - \lambda} + \frac{5 + 4 \csc \frac{1}{8}}{(1 - \lambda)\Gamma_q(5/2)}(b - a)_q^\alpha \right] \sum_{k=0}^{\infty} \left( \frac{5}{1 - \lambda} \Gamma_q\left(\frac{3}{2}\right) \right)^k \frac{(b - a)_q^{\frac{3}{2}k}}{\Gamma_q\left(\frac{3}{2}k + 1\right)} < \epsilon$$

时, 式 (4) 成立; 因此, 根据定理 3 知系统 (5) 是关于  $\{a, J, \delta, \epsilon, \tau\}$  有限时间稳定的.

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