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一类有序分数阶 q-差分系统边值问题解的存在性

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摘要：研究了一类有序分数阶 q-差分系统解的唯一性和存在性. 首先利用 q-指数函数给出了该方程解的表达式，然后分别利用 Leray-Schauder 选择定理、Krasnoselskii 不动点定理和 Banach 压缩映像原理证明了该系统解的存在性和唯一性.

关键词：有序分数阶 q-差分系统；不动点定理；解的存在性
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Existence of solutions for boundary value problems with a coupled system of sequential fractional q-differences

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Abstract: We study the existence and uniqueness of solutions for a class of the sequential fractional q-differences system. Firstly, using q-exponential, a representation for the solution to this equation is given. Then the existence and uniqueness of solutions are proven by using Leray-Schauder alternative theorem, Krasnoselskii fixed point theorem and Banach contraction mapping principle.

Keywords: sequential fractional q-differences system; fixed point theorem; existence of solutions

0　引言

1910 年，Jackson[1] 提出了 q-微积分的概念，之后由 Al-Salam[2] 和 Agarwal[3] 给出了分数阶 q-微积分的基本概念和性质. 近年来，q-差分微积分被广泛地应用于数学和工程科学，特别是在数学物理模型、动力系统、量子物理和经济学等方面发挥着重要作用，例如在金融市场中，将 q-差分的理论知识应用于股票收益率等问题[4-5]. 近年来，很多学者研究了分数阶 q-差分方程边值问题并取得了一些研究成果[6-11]，其中文献[11]讨论了有序分数阶方程：

$$\begin{cases} {}^{c}D_q^{\alpha}(D_q+\lambda)[y](x)=f(x,y(x)),\ 0\leqslant x\leqslant 1; \\ y(0)=D_q[y](0)=0,\ D_q[y](1)=\beta. \end{cases}$$

其中：$1<\alpha<2$，$0<\lambda<1$，$\beta>0$，$f\in C([0,1]\times\mathbf{R},\mathbf{R})$. 受文献[11]的启发，本文讨论如下有序分数阶 q-差分系统：

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$$
\begin{cases}
({}^cD_q^\alpha + k\,{}^cD_q^{\alpha-1})u(t) = f(t,u(t),v(t)),\ t\in[0,1];\\
({}^cD_q^\beta + k\,{}^cD_q^{\beta-1})v(t) = g(t,u(t),v(t)),\ t\in[0,1];\\
u(0)=D_qu(0)=0,\ u(1)=MD_qu(\xi);\\
v(0)=D_qv(0)=0,\ v(1)=ND_qv(\eta).
\end{cases}
\tag{1}
$$

这里：$2<\alpha$；$\beta\leqslant 3$；$0<q<1$；$\xi,\eta\in(0,1]$；$k>0$；M,N 是实数；$f,g:[0,1]\times\mathbf{R}\times\mathbf{R}\to\mathbf{R}$ 是连续函数. 本文将分别利用 Leray-Schauder 选择定理、Krasnoselskii 不动点定理和 Banach 压缩映像原理证明该方程解的存在性和唯一性.

1 预备知识

定义 1[11] $[a]_q=\dfrac{1-q^a}{1-q}$，$a\in\mathbf{R}$，$q\in(0,1)$.

定义 2[11] 幂指函数 $(a-b)^n$ 的 q-类似定义为：

$$(a-b)^{(0)}=1,\ (a-b)^{(n)}=\prod_{k=0}^{n-1}(a-bq^k),\ n\in\mathbf{N},\ a,b\in\mathbf{R};$$

$$(a-b)^{(\alpha)}=a^\alpha\prod_{n=0}^{\infty}\frac{a-bq^n}{a-bq^{\alpha+n}},\ \alpha\in\mathbf{R},\text{特别地},b=0\text{ 时 }a^{(\alpha)}=a^\alpha.$$

定义 3[11] q-Γ 函数定义为 $\Gamma_q(x)=\dfrac{(1-q)^{(x-1)}}{(1-q)^{x-1}}$，$x\in\mathbf{R}\backslash\{0,-1,-2,\cdots\}$，易知 $\Gamma_q(x+1)=[x]_q\Gamma_q(x)$.

定义 4[11] 函数 $f(t)$ 的 q-导数定义为：$D_qf(t)=\dfrac{f(t)-f(qt)}{(1-q)t}$，$D_qf(0)=\lim\limits_{t\to 0}D_qf(t)$. 函数 f 的高阶 q-导数定义为：$D_q^0f(t)=f(t)$，$D_q^nf(t)=D_q(D_q^{n-1}f)(t)$，$n\in\mathbf{N}$.

定义 5[11] 函数 $f(t)$ 在区间$[0,b]$上的 q-积分定义为：

$$I_qf(t)=\int_0^t f(s)\mathrm{d}_qs=t(1-q)\sum_{n=0}^{\infty}f(tq^n)q^n,\ t\in[0,b].$$

定义 6[11] Riemann-Liouville 型分数阶 q-积分定义为：

$$I_q^0f(t)=f(t),\ I_q^\alpha f(t)=\frac{1}{\Gamma_q(\alpha)}\int_0^t(t-qs)^{(\alpha-1)}f(s)\mathrm{d}_qs,\ \alpha>0,\ t\in[0,1].$$

Riemann-Liouville 型分数阶 q-导数定义为：

$$D_q^0f(t)=f(t),\ D_q^\alpha f(t)=(D_q^mI_q^{m-\alpha}f)(t),\ \alpha>0,\ t\in[0,1],$$

其中 $f(t)$ 是定义在$[0,1]$上的函数，m 是不小于 α 的最小整数. Caputo 型分数阶 q-导数定义为：

$${}^cD_q^0f(t)=f(t),\ {}^cD_q^\alpha f(t)=I_q^{\lceil\alpha\rceil-\alpha}(D_q^{\lceil\alpha\rceil}f)(t),\ \alpha>0,\ t\in[0,1],$$

其中 $f(t)$ 是定义在$[0,1]$上的函数，$\lceil\alpha\rceil$是不小于 α 的最小整数. 显然，当 $\alpha\in\mathbf{N}$ 时，${}^cD_q^\alpha f=D_q^\alpha f$. 特别地，$I_q^\alpha(t^{(\lambda)})=\dfrac{1}{\Gamma_q(\alpha)}\displaystyle\int_0^t(t-qs)^{(\alpha-1)}s^{(\lambda)}\mathrm{d}_qs=\dfrac{\Gamma_q(\lambda+1)t^{(\alpha+\lambda)}}{\Gamma_q(\alpha+\lambda+1)}$，$I_q^\alpha(1)(t)=\dfrac{t^{(\alpha)}}{\Gamma_q(\alpha+1)}$.

定义 7[11] 定义标准 q-指数函数为：

$$e_q^z=\sum_{n=0}^{\infty}\frac{z^n}{[n]!}=\prod_{k=0}^{\infty}(1-(1-q)q^kz)^{-1},\ E_q^z=\sum_{n=0}^{\infty}\frac{z^n}{[\tilde{n}]!}=\prod_{k=0}^{\infty}(1+(1-q)q^kz),$$

其中 $q>0$，z 是复数，$[n]!=[1][2]\cdots[n]$，$[k]=1+q+q^2+\cdots+q^{k-1}$，$[\tilde{n}]!=[\tilde{1}][\tilde{2}]\cdots[\tilde{n}]$，$[\tilde{k}]=1+\dfrac{1}{q}+\dfrac{1}{q^2}+\cdots+\dfrac{1}{q^{k-1}}$. 标准 q-指数函数满足如下性质：$e_q^zE_q^{-z}=1$，$D_qe_q^z=e_q^z$，$D_qE_q^z=E_q^{qz}$.

性质 1[11] 设 $\alpha\geqslant 0$，I 是包含原点的实区间且 $a,b\in I$，$f(t)$ 和 $g(t)$ 是定义在 I 到 $\mathbf{R}$ 上的函数，则：

(A_1) $(D_qI_qf)(x)=f(x)$，$(I_qD_qf)(t)=f(t)-f(0)$，$({}^cD_q^{\alpha}I_q^{\alpha}f)(t)=f(t)$；

(A_2) $[a(t-s)]^{(\alpha)}=a^{\alpha}(t-s)^{(\alpha)}$，${}_tD_q(t-s)^{(\alpha)}=[\alpha]_q(t-s)^{(\alpha-1)}$；

(A_3) $(D_qfg)(t)=(D_qf)(t)g(t)+f(qt)(D_qg)(t)$；

(A_4) $\int_a^b f(t)(D_qg)(t)\mathrm{d}_qt=[f(t)g(t)]_a^b-\int_a^b g(qt)(D_qf)(t)\mathrm{d}_qt$.

这里 ${}_iD_q$ 表示与变量 i 有关的 q-导数.

性质 2[11]　设 $\alpha,\beta\geqslant 0$，$f(x)$ 是定义在$[0,1]$上的函数，则$(I_q^{\beta}I_q^{\alpha}f)(x)=(I_q^{\alpha+\beta}f)(x)$.

性质 3[11]　设 $\alpha\geqslant 0$，则$(I_q^{\alpha}\,{}^cD_q^{\alpha}f)(t)=f(t)-\sum\limits_{n=0}^{\lceil\alpha\rceil-1}\dfrac{t^n}{\Gamma_q(n+1)}D_q^n(f)(0)$.

性质 4[6]　设 $\alpha\in\mathbf{R}^+\setminus\mathbf{N}_0$，则$(D_q^{\alpha+1}f)(x)=(D_q^{\alpha}D_qf)(x)$.

引理 1[11](Leray-Schauder 选择定理)　设 E 是 Banach 空间 $\mathscr{B}$ 的凸闭子集，开集 $U\subseteq E$，且 $0\in U$. 设 $T:\overline{U}\to E$ 是连续的紧算子，则下列二者之一成立：

(B_1) 存在一个 $x\in\overline{U}$，使得 $Tx=x$；

(B_2) 存在一个 $x\in\partial U$ 以及一个 $\lambda\in(0,1)$，有 $x=\lambda Tx$.

引理 2[11](Krasnoselskii 不动点定理)　设 K 是 Banach 空间 E 的有界凸闭子集，$T+S$ 在 K 内至少存在一个不动点，若 $T,S:K\to E$ 满足：

(C_1) 对任意 $x,y\in K$，有 $Tx+Sy\in K$；

(C_2) T 是压缩映像；

(C_3) S 在 K 上是全连续的.

2　主要结果及其证明

定理 1　假设 $h\in C[0,1]$，$(M-1)k-Mke_q^{-k\xi}-e_q^{-k}\neq-1$，则分数阶 q-差分方程

$$\begin{cases}({}^cD_q^{\alpha}+k\,{}^cD_q^{\alpha-1})u(t)=h(t);\\ u(0)=D_qu(0)=0,\ u(1)=MD_qu(\xi),\ \xi\in(0,1]\end{cases}\tag{2}$$

有唯一解

$$u(t)=e_q^{-kt}\int_0^t E_q^{kqs}I^{\alpha-1}h(s)\mathrm{d}_qs+(kt+e_q^{-kt}-1)K_h,\tag{3}$$

其中 $K_h=\dfrac{1}{\Delta_1}\left(e_q^{-k}\int_0^1 E_q^{kqs}I_q^{\alpha-1}h(s)\mathrm{d}_qs+kMe_q^{-k\xi}\int_0^{\xi}E_q^{kqs}I_q^{\alpha-1}h(s)\mathrm{d}_qs-MI_q^{\alpha-1}h(\xi)\right)$，$\Delta_1=(M-1)k-Mke_q^{-k\xi}-e_q^{-k}+1$.

证明　设 $u(t)$ 是方程(2)的解. 根据定义 6 和性质 3，可得：

$$I_q^{\alpha}\,{}^cD_q^{\alpha}u(t)=u(t)-a_0-a_1t-a_2t^2,\ I_q^{\alpha-1}\,{}^cD_q^{\alpha-1}u(t)=u(t)-b_1-b_2t.$$

根据性质 2 有 $I_q^{\alpha}\,{}^cD_q^{\alpha-1}u(t)=I_q^1I_q^{\alpha-1}\,{}^cD_q^{\alpha-1}u(t)=\int_0^t u(t)\mathrm{d}_qt-b_0-b_1t-\dfrac{b_1}{[2]_q}t^2$，所以由方程(2)有

$$u(t)+k\int_0^t u(s)\mathrm{d}_qs=I_q^{\alpha}h(t)+c_0+c_1t+\frac{c_2}{[2]_q}t^2,\ c_0,c_1,c_2\in\mathbf{R}.\tag{4}$$

将式(4)两边做微分 D_q 运算，可得

$$(D_q+k)u(t)=I_q^{\alpha-1}h(t)+c_1+c_2t.\tag{5}$$

由边值条件 $u(0)=D_qu(0)=0$ 得 $c_1=0$. 由于 $D_q(E_q^{kt}u(t))=kE_q^{kqt}u(t)+E_q^{kqt}D_qu(t)=E_q^{kqt}(D_q+k)u(t)$，所以有

$$D_q(E_q^{kt}u(t))=E_q^{kqt}(I_q^{\alpha-1}h(t)+c_2t).\tag{6}$$

将式(6)两边做积分 I_q 运算，可得 $E_q^{kt}u(t)=\int_0^t E_q^{kqs}I_q^{\alpha-1}h(s)\mathrm{d}_qs+c_2\int_0^t sE_q^{kqs}\mathrm{d}_qs$. 又由于

$$\int_0^t E_q^{kqs} s\,\mathrm{d}_q s=\frac{1}{k}\int_0^t sD_q(E_q^{ks})\,\mathrm{d}_q s=\frac{1}{k}\left(tE_q^{kt}-\int_0^t E_q^{kqs}\,\mathrm{d}_q s\right)=\frac{1}{k}\left(tE_q^{kt}-\frac{1}{k}\int_0^t D_q(E_q^{ks})\,\mathrm{d}_q s\right)=$$
$$\frac{1}{k}\left(tE_q^{kt}-\frac{1}{k}(E_q^{kt}-1)\right)=\frac{kE_q^{kt}t-E_q^{kt}+1}{k^2},$$

于是有 $E_q^{kt}u(t)=\int_0^t E_q^{kqs}I_q^{\alpha-1}h(s)\,\mathrm{d}_q s+\frac{c_2(kE_q^{kt}t-E_q^{kt}+1)}{k^2}$. 故有

$$u(t)=\frac{1}{E_q^{kt}}\int_0^t E_q^{kqs}I_q^{\alpha-1}h(s)\,\mathrm{d}_q s+\frac{c_2(kE_q^{kt}t-E_q^{kt}+1)}{k^2E_q^{kt}}=e_q^{-kt}\int_0^t E_q^{kqs}I_q^{\alpha-1}h(s)\,\mathrm{d}_q s+\frac{kt+e_q^{-kt}-1}{k^2}c_2. \quad (7)$$

将式(7)两边做微分 D_q 运算,可得 $D_qu(t)=I_q^{\alpha-1}h(t)-ke_q^{-kt}\int_0^t E_q^{kqs}I_q^{\alpha-1}h(s)\,\mathrm{d}_q s+\frac{(1-e_q^{-kt})}{k}c_2$. 由边值条件 $u(1)=MD_qu(\xi)$, 可得

$$c_2=\frac{k^2e_q^{-k}\int_0^1 E_q^{kqs}I_q^{\alpha-1}h(s)\,\mathrm{d}_q s+k^3Me_q^{-k\xi}\int_0^\xi E_q^{kqs}I_q^{\alpha-1}h(s)\,\mathrm{d}_q s-Mk^2I_q^{\alpha-1}h(\xi)}{(M-1)k-Mke_q^{-k\xi}-e_q^{-k}+1}. \quad (8)$$

将式(8)代入式(7)即可得式(3).

设空间 $U=C[0,1]$, 赋范数 $\|u\|=\max\limits_{0\leqslant t\leqslant 1}|u(t)|$, 显然 $(U,\|\cdot\|)$ 是 Banach 空间, 设空间 $W=((u,v)\mid(u,v)\in U\times U)$, 赋范数 $\|(u,v)\|=\|u\|+\|v\|$, 显然$(W,\|(u,v)\|)$也是Banach空间. 根据定理 1, 定义算子 $T:W\to W$ 如下:

$$T(u,v)(t)=(T_1(u,v)(t),T_2(u,v)(t)), \quad (9)$$

其中:

$$T_1(u,v)(t)=e_q^{-kt}\int_0^t E_q^{kqs}I_q^{\alpha-1}f(s,u(s),v(s))\,\mathrm{d}_q s+(kt+e_q^{-kt}-1)K_f;$$

$$T_2(u,v)(t)=e_q^{-kt}\int_0^t E_q^{kqs}I_q^{\beta-1}g(s,u(s),v(s))\,\mathrm{d}_q s+(kt+e_q^{-kt}-1)K_g.$$

这里

$$K_f=\frac{1}{\Delta_1}\Big(e_q^{-k}\int_0^1 E_q^{kqs}I_q^{\alpha-1}f(s,u(s),v(s))\,\mathrm{d}_q s+$$
$$kMe_q^{-k\xi}\int_0^\xi E_q^{kqs}I_q^{\alpha-1}f(s,u(s),v(s))\,\mathrm{d}_q s-MI_q^{\alpha-1}f(\xi,u(\xi),v(\xi))\Big);$$

$$K_g=\frac{1}{\Delta_2}\Big(e_q^{-k}\int_0^1 E_q^{kqs}I_q^{\beta-1}g(s,u(s),v(s))\,\mathrm{d}_q s+$$
$$kNe_q^{-k\eta}\int_0^\eta E_q^{kqs}I_q^{\beta-1}g(s,u(s),v(s))\,\mathrm{d}_q s-NI_q^{\beta-1}g(\eta,u(\eta),v(\eta))\Big).$$

$$\Delta_1=(M-1)k-Mke_q^{-k\xi}-e_q^{-k}+1;\ \Delta_2=(N-1)k-Nke_q^{-k\eta}-e_q^{-k}+1.$$

为了方便计算,记:

$$\omega_1=\frac{|\Delta_1|+(k+e_q^{-k}-1)(2k|M|\xi^{\alpha-1}+1)}{|\Delta_1|k\Gamma_q(\alpha)};\ \omega_2=\frac{|\Delta_2|+(k+e_q^{-k}-1)(2k|N|\eta^{\beta-1}+1)}{|\Delta_2|k\Gamma_q(\beta)};$$

$$\Omega_1=\frac{2(k+e_q^{-k}-1)}{|\Delta_1|}\left(e_q^{-k}\int_0^1 E_q^{kqs}I_q^{\alpha-1}p_1(s)\,\mathrm{d}_q s+k|M|e_q^{-k\xi}\int_0^\xi E_q^{kqs}I_q^{\alpha-1}p_1(s)\,\mathrm{d}_q s+|M|I_q^{\alpha-1}p_1(\xi)\right);$$

$$\Omega_2=\frac{2(k+e_q^{-k}-1)}{|\Delta_1|}\left(e_q^{-k}\int_0^1 E_q^{kqs}I_q^{\alpha-1}p_2(s)\,\mathrm{d}_q s+k|M|e_q^{-k\xi}\int_0^\xi E_q^{kqs}I_q^{\alpha-1}p_2(s)\,\mathrm{d}_q s+|M|I_q^{\alpha-1}p_2(\xi)\right);$$

$$\Omega_3=\frac{2(k+e_q^{-k}-1)}{|\Delta_2|}\left(e_q^{-k}\int_0^1 E_q^{kqs}I_q^{\beta-1}q_1(s)\,\mathrm{d}_q s+k|N|e_q^{-k\eta}\int_0^\eta E_q^{kqs}I_q^{\beta-1}q_1(s)\,\mathrm{d}_q s+|N|I_q^{\beta-1}q_1(\eta)\right);$$

$$\Omega_4=\frac{2(k+e_q^{-k}-1)}{|\Delta_2|}\left(e_q^{-k}\int_0^1 E_q^{kqs}I_q^{\beta-1}q_2(s)\,\mathrm{d}_q s+k|N|e_q^{-k\eta}\int_0^\eta E_q^{kqs}I_q^{\beta-1}q_2(s)\,\mathrm{d}_q s+|N|I_q^{\beta-1}q_2(\eta)\right);$$

$$\Omega=\max\{\Omega_1,\Omega_2,\Omega_3,\Omega_4\}.$$

首先，利用 Leray-Schauder 选择定理证明边值问题(1) 至少存在一个解.

定理 2　边值问题(1) 至少有一个解，假设如下(H_1) 和(H_2) 成立：

(H_1) 存在连续单调递增函数 $\psi_i,\phi_i:[0,\infty)\to[0,\infty)\ (i=1,2)$ 和连续函数 $G_i,H_i:[0,1]\to\mathbf{R}$ $(i=1,2,3)$，使得对 $\forall(t,u,v)\in[0,1]\times\mathbf{R}\times\mathbf{R}$ 有：$|f(t,u,v)|\leqslant G_1(t)\psi_1(|u|)+G_2(t)\psi_2(|v|)+G_3(t)$，$|g(t,u,v)|\leqslant H_1(t)\phi_1(|u|)+H_2(t)\phi_2(|v|)+H_3(t)$；

(H_2) 存在一个常数 $r>0$，使得 $r>(\|G_1\|\psi_1(r)+\|G_2\|\psi_2(r)+\|G_3\|)\omega_1+(\|H_1\|\phi_1(r)+\|H_2\|\phi_2(r)+\|H_3\|)\omega_2$.

证明　设 T 是式(9) 中定义的算子，由于 f,g 是连续函数，所以 T 是连续的. 为了应用引理 1，首先证明 T 在集合 $\bar{B}_r=\{(u,v)\in W\mid\|(u,v)\|\leqslant r\}$ 上一致有界，其中 r 为条件(H_2) 中的正常数. 事实上，$\forall(u,v)\in\bar{B}_r$ 和 $t\in[0,1]$，有

$$
\begin{aligned}
|T_1(u,v)(t)|\leqslant&\max_{0\leqslant t\leqslant 1}\Big\{e_q^{-kt}\int_0^t E_q^{kqs}\Big(\int_0^s\frac{(s-qm)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(m,u(m),v(m))|\mathrm{d}_qm\Big)\mathrm{d}_qs+\\
&\frac{kt+e_q^{-kt}-1}{|\Delta_1|}\Big(e_q^{-k}\int_0^1 E_q^{kqs}\Big(\int_0^s\frac{(s-qm)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(m,u(m),v(m))|\mathrm{d}_qm\Big)\mathrm{d}_qs+\\
&k|M|e_q^{-k\xi}\int_0^\xi E_q^{kqs}\Big(\int_0^s\frac{(s-qm)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(m,u(m),v(m))|\mathrm{d}_qm\Big)\mathrm{d}_qs+|M|\cdot\\
&\int_0^\xi\frac{(\xi-qm)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(m,u(m),v(m))|\mathrm{d}_qm\Big)\Big\}\leqslant\frac{\|G_1\|\psi_1(\|u\|)+\|G_2\|\psi_2(\|v\|)+\|G_3\|}{|\Delta_1|k\Gamma_q(\alpha)}\cdot\\
&\max_{0\leqslant t\leqslant 1}\{|\Delta_1|t^{\alpha-1}+(kt+e_q^{-kt}-1)(2k|M|\xi^{\alpha-1}+1)\}\leqslant\\
&\frac{\|G_1\|\psi_1(\|u\|)+\|G_2\|\psi_2(\|v\|)+\|G_3\|}{|\Delta_1|k\Gamma_q(\alpha)}[|\Delta_1|+(k+e_q^{-k}-1)(2k|M|\xi^{\alpha-1}+1)]\leqslant\\
&(\|G_1\|\psi_1(r)+\|G_2\|\psi_2(r)+\|G_3\|)\omega_1.
\end{aligned}
$$

同理可得

$$
\begin{aligned}
|T_2(u,v)(t)|\leqslant&\frac{\|H_1\|\phi_1(\|u\|)+\|H_2\|\phi_2(\|v\|)+\|H_3\|}{|\Delta_2|k\Gamma_q(\beta)}\cdot\\
&\max_{0\leqslant t\leqslant 1}\{|\Delta_2|t^{\beta-1}+(kt+e_q^{-kt}-1)(2k|N|\eta^{\alpha-1}+1)\}=\\
&\frac{\|H_1\|\phi_1(r)+\|H_2\|\phi_2(r)+\|H_3\|}{|\Delta_2|k\Gamma_q(\beta)}[|\Delta_2|+(k+e_q^{-k}-1)(2k|N|\eta^{\alpha-1}+1)]\leqslant\\
&(\|H_1\|\phi_1(r)+\|H_2\|\phi_2(r)+\|H_3\|)\omega_2.
\end{aligned}
$$

因此 $\|T(u,v)\|\leqslant(\|G_1\|\psi_1(r)+\|G_2\|\psi_2(r)+\|G_3\|)\omega_1+(\|H_1\|\phi_1(r)+\|H_2\|\phi_2(r)+\|H_3\|)\omega_2\leqslant r$，故 $T(\bar{B}_r)$ 是一致有界的.

其次证明 $T(\bar{B}_r)$ 是等度连续的. 事实上，$\forall t_1,t_2\in[0,1]$ 且 $t_1<t_2$，有

$$
\begin{aligned}
|T_1(u,v)(t_2)-T_1(u,v)(t_1)|=&(e_q^{-kt_2}-e_q^{-kt_1})\int_0^{t_2}E_q^{kqs}I_q^{\alpha-1}|f(s,u(s),v(s))|\mathrm{d}_qs+\\
&e_q^{-kt_1}\int_{t_1}^{t_2}E_q^{kqs}I_q^{\alpha-1}|f(s,u(s),v(s))|\mathrm{d}_qs+[(kt_2-kt_1)+(e_q^{-kt_2}-e_q^{-kt_1})]K_f\to 0\ (t_1\to t_2).
\end{aligned}
$$

同理可得 $|T_2(u,v)(t_2)-T_2(u,v)(t_1)|\to 0\ (t_1\to t_2)$，于是有 $|T(u,v)(t_2)-T(u,v)(t_1)|\to 0\ (t_1\to t_2)$，故 $T(\bar{B}_r)$ 是等度连续的. 根据 Arzela-Ascoli 定理可知 $\bar{B}_r$ 是紧的，由此知 $T(u,v)$ 是全连续的.

最后证明引理 1 中的条件(B_2) 不成立. 假设条件(B_2) 成立，则存在 $\lambda\in(0,1)$，且 $(u,v)\in\partial B_r$，使得 $(u,v)=\lambda T(u,v)$. 故对于 $\forall t\in[0,1]$，有 $u(t)=\lambda T_1(u,v)(t)$，$v(t)=\lambda T_2(u,v)(t)$，因此有 $\|(u,v)\|=r$，且

$$
\begin{aligned}
|u(t)|=\lambda|T_1(u,v)(t)|<&\\
&\frac{\|G_1\|\psi_1(r)+\|G_2\|\psi_2(r)+\|G_3\|}{|\Delta_1|k\Gamma_q(\alpha)}[|\Delta_1|+(k+e_q^{-k}-1)(2k|M|\xi^{\alpha-1}+1)],
\end{aligned}
$$

$$|v(t)|=\lambda|T_2(u,v)(t)|<$$
$$\frac{\|H_1\|\phi_1(r)+\|H_2\|\phi_2(r)+\|H_3\|}{|\Delta_2|k\Gamma_q(\beta)}[|\Delta_2|+(k+e_q^{-k}-1)(2k|N|\eta^{\alpha-1}+1)].$$

即 $\|u\|<(\|G_1\|\psi_1(r)+\|G_2\|\psi_2(r)+\|G_3\|)\omega_1$, $\|v\|<(\|H_1\|\phi_1(r)+\|H_2\|\phi_2(r)+\|H_3\|)\omega_2$. 因此有 $\|(u,v)\|<(\|G_1\|\psi_1(r)+\|G_2\|\psi_2(r)+\|G_3\|)\omega_1+(\|H_1\|\phi_1(r)+\|H_2\|\phi_2(r)+\|H_3\|)\omega_2$. 故有 $r<(\|G_1\|\psi_1(r)+\|G_2\|\psi_2(r)+\|G_3\|)\omega_1+(\|H_1\|\phi_1(r)+\|H_2\|\phi_2(r)+\|H_3\|)\omega_2$, 这与($H_2$)矛盾,所以由引理 1 可得边值问题(1)至少有一个解.

其次,利用 Krasnoselskii 不动点定理证明边值问题(1)至少存在一个解.

定理 3 边值问题(1)至少有一个解,假设如下(H_3)—(H_5)成立:

(H_3) 存在 q-可积函数 $p_i,q_i:[0,1]\to[0,\infty)(i=1,2)$,使得 $\forall t\in[0,1]$, $\forall u_1,u_2,v_1,v_2\in\mathbf{R}$,有:

$$|f(t,u_1,v_1)-f(t,u_2,v_2)|\leqslant p_1(t)|u_1-u_2|+p_2(t)|v_1-v_2|;$$
$$|g(t,u_1,v_1)-g(t,u_2,v_2)|\leqslant q_1(t)|u_1-u_2|+q_2(t)|v_1-v_2|;$$

(H_4) 存在连续函数 $\rho_1,\rho_2:[0,1]\to[0,\infty)$, 使得 $\forall t\in[0,1]$, $\forall u,v\in\mathbf{R}$, 有: $|f(t,u,v)|\leqslant\rho_1(t)$, $|g(t,u,v)|\leqslant\rho_2(t)$;

(H_5) $\Omega<1$.

证明 为了应用引理 2,定义函数:

$$P_1(u,v)(t)=e_q^{-kt}\int_0^t E_q^{kqs}\left(\int_0^s\frac{(s-qm)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}f(m,u(m),v(m))\mathrm{d}_qm\right)\mathrm{d}_qs,$$
$$P_2(u,v)(t)=(kt+e_q^{-kt}-1)K_f, \tag{10}$$
$$Q_1(u,v)(t)=e_q^{-kt}\int_0^t E_q^{kqs}\left(\int_0^s\frac{(s-qm)^{(\beta-2)}}{\Gamma_q(\beta-1)}g(m,u(m),v(m))\mathrm{d}_qm\right)\mathrm{d}_qs,$$
$$Q_2(u,v)(t)=(kt+e_q^{-kt}-1)K_g. \tag{11}$$

显然

$$T_1(u,v)(t)=P_1(u,v)(t)+P_2(u,v)(t),\ T_2(u,v)(t)=Q_1(u,v)(t)+Q_2(u,v)(t).$$

令 $\max\limits_{0\leqslant t\leqslant1}|\rho_1(t)|=\|\rho_1\|$, $\max\limits_{0\leqslant t\leqslant1}|\rho_2(t)|=\|\rho_2\|$,定义集合 $B_{r_1}=\{(u,v)\in W\mid\|(u,v)\|\leqslant r_1\}$,其中 $r_1=\max\left\{\frac{2\|\rho_1\|}{|\Delta_1|k\Gamma_q(\alpha)}(\Delta_1+(k+e_q^{-k}-1)(2k|M|\xi^{\alpha-1})),\frac{2\|\rho_2\|}{|\Delta_2|k\Gamma_q(\beta)}(\Delta_2+(k+e_q^{-k}-1)(2k|N|\eta^{\beta-1}))\right\}$.
这里通过证明 T_1 在集合 B_{r_1} 上的性质,类似地可得 T_2 在集合 B_{r_1} 上也具有相同的性质.

首先证明算子 $P_1(u_1,v_1)+P_2(u_2,v_2)$ 满足引理 2 中的(C_1). 注意到 $\forall(u_1,v_1),(u_2,v_2)\in B_{r_1}$,有

$$\|P_1(u_1,v_1)+P_2(u_2,v_2)\|\leqslant$$
$$\max_{0\leqslant t\leqslant1}\left\{\frac{e_q^{-kt}}{\Gamma_q(\alpha-1)}\int_0^t E_q^{kqs}\left(\int_0^s(s-qm)^{(\alpha-2)}|f(m,u_1(m),v_1(m))|\mathrm{d}_qm\right)\mathrm{d}_qs+\right.$$
$$\frac{kt+e_q^{-kt}-1}{|\Delta_1|\Gamma_q(\alpha-1)}\left(e_q^{-k}\int_0^1 E_q^{kqs}\left(\int_0^s(s-qm)^{(\alpha-2)}|f(m,u_2(m),v_2(m))|\mathrm{d}_qm\right)\mathrm{d}_qs+\right.$$
$$k|M|e_q^{-k\xi}\int_0^\xi E_q^{kqs}\left(\int_0^s(s-qm)^{(\alpha-2)}|f(m,u_2(m),v_2(m))|\mathrm{d}_qm\right)\mathrm{d}_qs+$$
$$\left.\left.|M|\int_0^\xi(\xi-qm)^{(\alpha-2)}|f(m,u_2(m),v_2(m))|\mathrm{d}_qm\right)\right\}\leqslant$$
$$\frac{\|\rho_1\|}{|\Delta_1|k\Gamma_q(\alpha)}(|\Delta_1|+(k+e_q^{-k}-1)(1+2k|M|\xi^{\alpha-1}))\leqslant\frac{r_1}{2}.$$

同理 $\|Q_1(u_1,v_1)+Q_2(u_2,v_2)\|\leqslant\frac{r_1}{2}$.

其次证明算子 P_2 是压缩映射. 事实上, $\forall(u_1,v_1),(u_2,v_2)\in B_{r_1}$,有

$$\| P_2(u_1,v_1)-P_2(u_2,v_2)\| \leqslant \max_{0\leqslant t\leqslant 1}\left\{\frac{kt+e_q^{-kt}-1}{|\Delta_1|}\left(e_q^{-k}\int_0^1 E_q^{kqs}\cdot\right.\right.$$
$$\left(\int_0^s \frac{(s-qm)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(m,u_1(m),v_1(m))-f(m,u_2(m),v_2(m))|\mathrm{d}_q m\right)\mathrm{d}_q s+$$
$$k|M|e_q^{-k\xi}\int_0^\xi E_q^{kqs}\left(\int_0^s \frac{(s-qm)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(m,u_1(m),v_1(m))-f(m,u_2(m),v_2(m))|\mathrm{d}_q m\right)\mathrm{d}_q s+$$
$$\left.\left.|M|\int_0^\xi \frac{(\xi-qm)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(m,u_1(m),v_1(m))-f(m,u_2(m),v_2(m))|\mathrm{d}_q m\right)\right\}\leqslant$$
$$\frac{\|u_1-u_2\|(k+e_q^{-k}-1)}{|\Delta_1|}\left(e_q^{-k}\int_0^1 E_q^{kqs}I_q^{\alpha-1}p_1(s)\mathrm{d}_q s+k|M|e_q^{-k\xi}\int_0^\xi E_q^{kqs}I_q^{\alpha-1}p_1(s)\mathrm{d}_q s+|M|I_q^{\alpha-1}p_1(\xi)\right)+$$
$$\frac{\|v_1-v_2\|(k+e_q^{-k}-1)}{|\Delta_1|}\left(e_q^{-k}\int_0^1 E_q^{kqs}I_q^{\alpha-1}p_2(s)\mathrm{d}_q s+k|M|e_q^{-k\xi}\int_0^\xi E_q^{kqs}I_q^{\alpha-1}p_2(s)\mathrm{d}_q s+|M|I_q^{\alpha-1}p_2(\xi)\right)\leqslant$$
$$\frac{\Omega}{2}(\|u_1-u_2\|+\|v_1-v_2\|).$$

同理 $\|Q_2(u_1,v_1)-Q_2(u_2,v_2)\|\leqslant\dfrac{\Omega}{2}(\|u_1-u_2\|+\|v_1-v_2\|)$.

再次证明 $P_1(u,v)$ 是全连续.由于 f 是连续的,所以 P_1 是连续的.设 $(u,v)\in B_{r_1}$,则

$$\|P_1(u,v)\|=\max_{0\leqslant t\leqslant 1}\left\{e_q^{-kt}\int_0^t E_q^{kqs}\left(\int_0^s \frac{(s-qm)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(m,u(m),v(m))|\mathrm{d}_q m\right)\mathrm{d}_q s\right\}\leqslant$$
$$\max_{0\leqslant t\leqslant 1}\left\{e_q^{-kt}\int_0^t E_q^{kqs}\left(\int_0^s \frac{(s-qm)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\rho_1(t)\mathrm{d}_q m\right)\mathrm{d}_q s\right\}\leqslant\frac{\|\rho_1\|}{k\Gamma_q(\alpha)}.$$

故有 $P_1(B_{r_1})$ 一致有界.

这里证明 $T(B_{r_1})$ 是等度连续的.事实上,$\forall t_1,t_2\in[0,1]$ 且 $t_1<t_2$,有

$$|P_1(u,v)(t_2)-P_1(u,v)(t_1)|=(e_q^{-kt_2}-e_q^{-kt_1})\int_0^{t_2}E_q^{kqs}I_q^{\alpha-1}|f(s,u(s),v(s))|\mathrm{d}_q s+$$
$$e_q^{-kt_1}\int_{t_1}^{t_2}E_q^{kqs}I_q^{\alpha-1}|f(s,u(s),v(s))|\mathrm{d}_q s\to 0\ (t_1\to t_2),$$

因此 $P_1(B_{r_1})$ 是等度连续的.根据 Arzela-Ascoli 定理可知 $P_1(B_{r_1})$ 是紧的,由此知 $P_1(u,v)$ 是全连续.

最后证明假设中的所有条件都满足引理 2.定义算子:

$$P(u,v)(t)=(P_1(u,v)(t),Q_1(u,v)(t)),\ Q(u,v)(t)=(P_2(u,v)(t),Q_2(u,v)(t)).\tag{12}$$

因此有

$$T(u,v)(t)=(T_1(u,v)(t),T_2(u,v)(t))=$$
$$(P_1(u,v)(t)+P_2(u,v)(t),Q_1(u,v)(t)+Q_2(u,v)(t))=$$
$$(P_1(u,v)(t),Q_1(u,v)(t))+(P_2(u,v)(t),Q_2(u,v)(t))=P(u,v)(t)+Q(u,v)(t).$$

综上所述,有:

(i) $\forall(u_1,v_1),(u_2,v_2)\in B_{r_1}$,有 $\|P(u,v)+Q(u,v)\|=\|P_1(u,v)+P_2(u,v)\|+\|Q_1(u,v)+Q_2(u,v)\|\leqslant r_1$.因此 $P(u,v)+Q(u,v)\in B_{r_1}$,满足引理 2 中的(C_1).

(ii) $\forall t\in[0,1]$,$\forall(u_1,v_1),(u_2,v_2)\in B_{r_1}$,有 $\|Q(u_1,v_1)-Q(u_2,v_2)\|=\|P_2(u_1,v_1)-P_2(u_2,v_2)\|+\|Q_2(u_1,v_1)-Q_2(u_2,v_2)\|\leqslant\Omega(\|u_1-u_2\|+\|v_1-v_2\|)$.所以算子 Q 是压缩映射,满足引理 2 中的(C_2).

(iii) 由于 $P_1(B_{r_1})$ 和 $Q_1(B_{r_1})$ 一致有界,所以 $P(B_{r_1})$ 也一致有界.这是因为

$$\|P(u,v)\|=\|P_1(u,v)\|+\|Q_1(u,v)\|\leqslant\frac{\|\rho_1\|}{k\Gamma_q(\alpha)}+\frac{\|\rho_2\|}{k\Gamma_q(\beta)}.$$

再设 $\forall t_1,t_2\in[0,1]$ 且 $t_1<t_2$,则 $\forall(u,v)\in B_{r_1}$,有

$$|P(u,v)(t_2)-P(u,v)(t_1)|=|P_1(u,v)(t_2)-P_1(u,v)(t_1)|+$$

$$|Q_1(u,v)(t_2)-Q_1(u,v)(t_1)|\leqslant(e_q^{-kt_2}-e_q^{-kt_1})\int_0^{t_2}E_q^{kqs}\left(\int_0^s\frac{(s-qm)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\rho_1(m)\mathrm{d}_qm\right)\mathrm{d}_qs+e_q^{-kt_1}\cdot$$
$$\int_{t_1}^{t_2}E_q^{kqs}\left(\int_0^s\frac{(s-qm)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\rho_1(m)\mathrm{d}_qm\right)\mathrm{d}_qs+(e_q^{-kt_2}-e_q^{-kt_1})\int_0^{t_2}E_q^{kqs}\left(\int_0^s\frac{(s-qm)^{(\beta-2)}}{\Gamma_q(\beta-1)}\rho_2(m)\mathrm{d}_qm\right)\mathrm{d}_qs+$$
$$e_q^{-kt_1}\int_{t_1}^{t_2}E_q^{kqs}\left(\int_0^s\frac{(s-qm)^{(\beta-2)}}{\Gamma_q(\beta-1)}\rho_2(m)\mathrm{d}_qm\right)\mathrm{d}_qs\to0\ (t_1\to t_2).$$

因此 $P(B_{r_1})$ 是等度连续的,根据 Arzela-Ascoli 定理知 B_{r_1} 是紧的,由此可知 $P(u,v)$ 是全连续的,且满足引理 2 中的(C_3).综上,由引理 2 知边值问题(1) 至少有一个解.

最后,利用 Banach 压缩映像原理证明边值问题(1) 存在唯一解.

定理 4 假设(H_3)成立,且 $e_q^{-k}\int_0^1E_q^{kqs}[I_q^{\alpha-1}(p_1(s)+p_2(s))+I_q^{\beta-1}(q_1(s)+q_2(s))]\mathrm{d}_qs+2\Omega<1$,则边值问题(1) 有唯一解.

证明 设 $\max\limits_{t\in[0,1]}f(t,0,0)=K<+\infty$, $\max\limits_{t\in[0,1]}g(t,0,0)=L<+\infty$, $B_R=\{(u,v)\in W\mid\|(u,v)\|\leqslant R\}$.

这里取 $R\geqslant\dfrac{K\omega_1+L\omega_2}{1-e_q^{-k}\int_0^1E_q^{kqs}[I_q^{\alpha-1}(p_1(s)+p_2(s))+I_q^{\beta-1}(q_1(s)+q_2(s))]\mathrm{d}_qs-2\Omega}$.

首先证明 $TB_R\subset B_R$. $\forall(u,v)\in B_R$,有

$$|T_1(u,v)(t)|\leqslant\max_{0\leqslant t\leqslant1}\Bigg\{e_q^{-kt}\int_0^tE_q^{kqs}\left(\int_0^s\frac{(s-qm)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}(|f(m,u(m),v(m))-f(m,0,0)|+\right.$$
$$|f(m,0,0)|)\mathrm{d}_qm\Big)\mathrm{d}_qs+\frac{kt+e_q^{-kt}-1}{|\Delta_1|}\Bigg(e_q^{-k}\int_0^1E_q^{kqs}\left(\int_0^s\frac{(s-qm)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}(|f(m,u(m),v(m))-f(m,0,0)|+\right.$$
$$|f(m,0,0)|)\mathrm{d}_qm\Big)\mathrm{d}_qs+k|M|e_q^{-k\xi}\int_0^\xi E_q^{kqs}\left(\int_0^s\frac{(s-qm)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}(|f(m,u(m),v(m))-f(m,0,0)|+\right.$$
$$|f(m,0,0)|)\mathrm{d}_qm\Big)\mathrm{d}_qs+|M|\int_0^\xi\frac{(\xi-qm)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}(|f(m,u(m),v(m))-f(m,0,0)|+$$
$$|f(m,0,0)|)\mathrm{d}_qm\Bigg)\Bigg\}\leqslant\max_{0\leqslant t\leqslant1}\Bigg\{e_q^{-kt}\int_0^tE_q^{kqs}\left(\int_0^s\frac{(s-qm)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}(p_1(m)|u(m)|+p_2(m)|v(m)|+\right.$$
$$K)\mathrm{d}_qm\Big)\mathrm{d}_qs+\frac{kt+e_q^{-kt}-1}{|\Delta_1|}\Bigg(e_q^{-k}\int_0^1E_q^{kqs}\left(\int_0^s\frac{(s-qm)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}(p_1(m)|u(m)|+p_2(m)|v(m)|+\right.$$
$$K)\mathrm{d}_qm\Big)\mathrm{d}_qs+k|M|e_q^{-k\xi}\int_0^\xi E_q^{kqs}\left(\int_0^s\frac{(s-qm)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}(p_1(m)|u(m)|+p_2(m)|v(m)|+K)\mathrm{d}_qm\right)\mathrm{d}_qs+$$
$$|M|\int_0^\xi\frac{(\xi-qm)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}(p_1(m)|u(m)|+p_2(m)|v(m)|+K)\mathrm{d}_qm\Bigg)\Bigg\}\leqslant$$
$$\|u\|\max_{0\leqslant t\leqslant1}\Bigg\{e_q^{-kt}\int_0^tE_q^{kqs}I_q^{\alpha-1}p_1(s)\mathrm{d}_qs+\frac{kt+e_q^{-kt}-1}{|\Delta_1|}\Bigg(e_q^{-k}\int_0^1E_q^{kqs}I_q^{\alpha-1}p_1(s)\mathrm{d}_qs+$$
$$k|M|e_q^{-k\xi}\int_0^\xi E_q^{kqs}I_q^{\alpha-1}p_1(s)\mathrm{d}_qs+|M|I_q^{\alpha-1}p_1(\xi)\Bigg)\Bigg\}+\|v\|\max_{0\leqslant t\leqslant1}\Bigg\{e_q^{-kt}\int_0^tE_q^{kqs}I_q^{\alpha-1}p_2(s)\mathrm{d}_qs+$$
$$\frac{kt+e_q^{-kt}-1}{|\Delta_1|}\Bigg(e_q^{-k}\int_0^1E_q^{kqs}I_q^{\alpha-1}p_2(s)\mathrm{d}_qs+k|M|e_q^{-k\xi}\int_0^\xi E_q^{kqs}I_q^{\alpha-1}p_2(s)\mathrm{d}_qs+|M|I_q^{\alpha-1}p_2(\xi)\Bigg)\Bigg\}+$$
$$K\max_{0\leqslant t\leqslant1}\Bigg\{e_q^{-kt}\int_0^tE_q^{kqs}\left(\int_0^s\frac{(s-qm)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\mathrm{d}_qm\right)\mathrm{d}_qs+\frac{kt+e_q^{-kt}-1}{|\Delta_1|}\Bigg(e_q^{-k}\int_0^1E_q^{kqs}\left(\int_0^s\frac{(s-qm)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\mathrm{d}_qm\right)\mathrm{d}_qs+$$
$$k|M|e_q^{-k\xi}\int_0^\xi E_q^{kqs}\left(\int_0^s\frac{(s-qm)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\mathrm{d}_qm\right)\mathrm{d}_qs+|M|\int_0^\xi\frac{(\xi-qm)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\mathrm{d}_qm\Bigg)\Bigg\}\leqslant$$
$$\left(e_q^{-k}\int_0^1E_q^{kqs}I_q^{\alpha-1}(p_1(s)+p_2(s))\mathrm{d}_qs+\Omega\right)R+K\omega_1.$$

同理可得 $|T_2(u,v)(t)|\leqslant\left(e_q^{-k}\int_0^1E_q^{kqs}I_q^{\beta-1}(q_1(s)+q_2(s))\mathrm{d}_qs+\Omega\right)R+L\omega_2$,因此有 $\|T(u,v)\|\leqslant R$.

下面证明算子 T 是一个压缩映射. 事实上,$\forall(u_1,v_1),(u_2,v_2)\in W$,有

$$
\begin{aligned}
&|T_1(u_1,v_1)(t)-T_1(u_2,v_2)(t)|\leqslant \max_{0\leqslant t\leqslant 1}\Bigg\{e_q^{-kt}\int_0^t E_q^{kqs}\Big(\int_0^s \frac{(s-qm)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\cdot \\
&\quad |f(m,u_1(m),v_1(m))-f(m,u_2(m),v_2(m))|\,\mathrm{d}_q m\Big)\mathrm{d}_q s+\frac{kt+e_q^{-kt}-1}{|\Delta_1|}\cdot \\
&\quad\Bigg(e_q^{-k}\int_0^1 E_q^{kqs}\Big(\int_0^s \frac{(s-qm)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(m,u_1(m),v_1(m))-f(m,u_2(m),v_2(m))|\,\mathrm{d}_q m\Big)\mathrm{d}_q s+ \\
&\quad k|M|e_q^{-k\xi}\int_0^\xi E_q^{kqs}\Big(\int_0^s \frac{(s-qm)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(m,u_1(m),v_1(m))-f(m,u_2(m),v_2(m))|\,\mathrm{d}_q m\Big)\mathrm{d}_q s+ \\
&\quad |M|\Big|\int_0^\xi \frac{(\xi-qm)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(m,u_1(m),v_1(m))-f(m,u_2(m),v_2(m))|\,\mathrm{d}_q m\Bigg)\Bigg\}\leqslant \\
&\quad\Big(e_q^{-k}\int_0^1 E_q^{kqs}I_q^{\alpha-1}p_1(s)\,\mathrm{d}_q s+\frac{\Omega_1}{2}\Big)\|u_1-u_2\|+\Big(e_q^{-k}\int_0^1 E_q^{kqs}I_q^{\alpha-1}p_2(s)\,\mathrm{d}_q s+\frac{\Omega_2}{2}\Big)\|v_1-v_2\|\leqslant \\
&\quad\Big(e_q^{-k}\int_0^1 E_q^{kqs}I_q^{\alpha-1}(p_1(s)+p_2(s))\,\mathrm{d}_q s+\Omega\Big)(\|u_1-u_2\|+\|v_1-v_2\|).
\end{aligned}
$$

同理可得 $|T_2(u_1,v_1)(t)-T_2(u_2,v_2)(t)|\leqslant(e_q^{-k}\int_0^1 E_q^{kqs}I_q^{\beta-1}(q_1(s)+q_2(s))\mathrm{d}_q s+\Omega)(\|u_1-u_2\|+\|v_1-v_2\|)$,因此有 $\|T(u_1,v_1)-T(u_2,v_2)\|\leqslant(e_q^{-k}\int_0^1 E_q^{kqs}[I_q^{\alpha-1}(p_1(s)+p_2(s))+I_q^{\beta-1}(q_1(s)+q_2(s))]\mathrm{d}_q s+2\Omega)(\|u_1-u_2\|+\|v_1-v_2\|)$. 因此当 $e_q^{-k}\int_0^1 E_q^{kqs}[I_q^{\alpha-1}(p_1(s)+p_2(s))+I_q^{\beta-1}(q_1(s)+q_2(s))]\mathrm{d}_q s+2\Omega<1$ 时,算子 T 是一个压缩映射. 根据 Banach 压缩映像原理知,边值问题(1) 存在唯一解.

特别地,$\forall t\in[0,1]$,当(H_3)中的函数 $p_1(s)=K_1$, $p_2(s)=K_2$, $q_1(s)=L_1$, $q_2(s)=L_2$ 时,其中 K_i, $L_i(i=1,2)$ 为常数,可将定理 4 中的条件 $e_q^{-k}\int_0^1 E_q^{kqs}[I_q^{\alpha-1}(p_1(s)+p_2(s))+I_q^{\beta-1}(q_1(s)+q_2(s))]\mathrm{d}_q s+2\Omega<1$ 改为 $K_1\omega_1+L_1\omega_2+K_2\omega_1+L_2\omega_2<1$.

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