文章编号: 1004-4353(2017)01-0001-06

一类非线性分数阶 q-对称差分方程边值问题正解的存在性

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摘要: 考虑了一类 Riemann-Liouville 型非线性分数阶 q-对称差分方程边值问题正解的存在性. 首先分析了格林函数的一些性质,然后利用锥上的不动点定理证明了该方程正解的存在性,最后通过实例验证了本文所得结论的正确性.

关键词: q-对称 Riemann-Liouville 分数阶导数; 锥; 边值问题; 解的存在性
中图分类号: O175.6 **文献标识码:** A

Existence of positive solutions for boundary value problems of a class of nonlinear fractional q-symmetry differences equation

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Abstract: We consider the existence of positive solutions for boundary value problems of a class of nonlinear fractional q-symmetry differences equation. Firstly, some properties of the Green function are analyzed. The second, the existence of positive solutions for the boundary value problems is investigated by applying a fixed point theorem in cones. The end, we give an example to illustrate our results.

Keywords: q-symmetry Riemann-Liouville fractional order derivative; cones; boundary value problem; existence of solutions

0 引言

q-微积分概念由 Jackson 于 1910 年提出[1-2],之后 Al-Salam[3] 和 Agarwal[4] 给出了分数阶 q-微积分的基本概念和性质,丰富了 q-差分微积分的基本理论. 由于分数阶微积分可以处理含有任意阶导数的问题,因此它在许多领域得到了广泛应用,其相关研究也取得了一些成果[5-6]. 近年来,许多学者对分数阶 q-微积分边值问题也进行了研究,例如: R.Ferreira 在文献[8-9]中分别在条件 $\alpha\in(1,2]$, $u(0)=u(1)=0$ 和 $\alpha\in(2,3]$, $u(0)=D_qu(0)=0$, $D_qu(1)=\beta$ 下,利用锥上的不动点定理研究了分数阶 q-差分方程 $(D_q^\alpha u)(t)+f(t,u(t))=0$, $0\leqslant t\leqslant 1$ 解的存在性; Liang 等[10] 研究了分数阶 q-差分方程 $(D_q^\alpha u)(t)+f(t,u(t))=0$, $0\leqslant t\leqslant 1$ 的三点边值问题; Zhao 等[11] 研究了带有分数阶 q-积分边界条件的分数阶 q-差分方程 $(D_q^\alpha u)(t)+f(t,u(t))=0$, $0\leqslant t\leqslant 1$ 的正解的存在性,给出了该边值问题相应格林函数的性质,并利用压缩映像原理、单调迭代方法以及 Krasnoselskii 不动点定理证明了正解的存在性. 其他相关成果还可参阅文献[12-15]. 目前为止,相关分数阶 q-对称微积分的研究鲜有报道,2016 年 Sun 等在文献

收稿日期: 2016-12-24 **基金项目:** 国家自然科学基金资助项目(11161049)
* **通信作者:** 侯成敏(1963—),女,教授,研究方向为微分方程理论及其应用.

[16]中引入了分数阶q-对称积分和导数的概念,并证明了它们的一些性质.基于文献[16]的研究结果,本文将研究一类非线性分数阶q-对称差分方程边值问题正解的存在性.

1 预备知识

定义1[16] $\widetilde{[a]}_q=\dfrac{1-q^{2a}}{1-q^2}\,(a\in\mathbf{R})$, $\widetilde{[0]}_q!\ =1$, $\widetilde{[n]}_q!\ =\widetilde{[n]}_q\,\widetilde{[n-1]}_q\cdots\widetilde{[1]}_q$.

定义2[16] 幂指函数$(a-b)^n$的q-对称类似定义为:

$$\widetilde{(a-b)}^{(0)}=1;\ \widetilde{(a-b)}^{(k)}=\prod_{i=0}^{k-1}(a-bq^{2i+1}),\ k\in\mathbf{N},\ a,b\in\mathbf{R};$$

$$\widetilde{(a-b)}^{(\alpha)}=a^{\alpha}\frac{\prod\limits_{i=0}^{\infty}\left(1-\dfrac{b}{a}q^{2i+1}\right)}{\prod\limits_{i=0}^{\infty}\left(1-\dfrac{b}{a}q^{2(i+\alpha)+1}\right)},\ \alpha\in\mathbf{R},\ a\neq0.$$

特别地,$b=0$时$\tilde{a}^{(\alpha)}=a^{\alpha}$, $a=0$时$\widetilde{(a-b)}^{(\alpha)}=0$.

定义3[16] 函数$x(t)$的q-对称导数定义为:$(\widetilde{D}_qx)(t)=\dfrac{x(qt)-x(q^{-1}t)}{(q-q^{-1})t}$, $(\widetilde{D}_qx)(0)=x'(0)$,函数$x(t)$的高阶$q$-对称导数定义为:$(\widetilde{D}^0_{q,0}x)(t)=x(t)$, $(\widetilde{D}^n_{q,0}x)(t)=(\widetilde{D}_{q,0}\widetilde{D}^{n-1}_{q,0}x)(t)$, $n\in\mathbf{N}$.

定义4[16] q-对称Γ函数定义为$\widetilde{\Gamma}_q(x)=\widetilde{(1-q)}^{(x-1)}(1-q^2)^{1-x}\ (x\in\mathbf{R}/\{0,-1,-2,\cdots\})$. $x\in\mathbf{R}\backslash\{0,-1,-2,\cdots\}$. 易知$\widetilde{\Gamma}_q(1)=1$, $\widetilde{\Gamma}_q(x+1)=\widetilde{[x]}_q\widetilde{\Gamma}_q(x)$.

定义5[16] 函数$x(t)$的q-对称积分定义为:

$$(\widetilde{I}_{q,0}x)(t)=\int_0^t x(s)\tilde{d}_qs=t(1-q^2)\sum_{k=0}^{\infty}q^{2k}x(tq^{2k+1}),\ (\widetilde{I}_{q,a}x)(t)=\int_0^t x(s)\tilde{d}_qs-\int_0^a x(s)\tilde{d}_qs.$$

函数$x(t)$的高阶q-对称积分定义为:

$$(\widetilde{I}^0_{q,0}x)(t)=x(t),\ (\widetilde{I}^n_{q,0}x)(t)=(\widetilde{I}_{q,0}\widetilde{I}^{n-1}_{q,0}x)(t),\ n\in\mathbf{N}.$$

定义6[16] 函数$x(t)$的q-对称分数阶积分定义为

$$(\widetilde{I}^{\alpha}_{q,0}x)(t)=\frac{1}{\widetilde{\Gamma}_q(\alpha)}q^{\binom{\alpha}{2}}\int_0^t\widetilde{(t-s)}^{(\alpha-1)}x(q^{\alpha-1}s)\tilde{d}_qs,$$

其中$\dbinom{\alpha}{k}=\dfrac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)}\ (k\in\mathbf{N})$.

定义7[16] 函数$x(t)$的q-对称Riemann-Liouville分数阶导数定义为

$$(\widetilde{D}^{\alpha}_{q,0}x)(t)=\begin{cases}(\widetilde{I}^{-\alpha}_{q,0}x)(t),\ \alpha<0;\\ x(t),\ \alpha=0;\\ (\widetilde{D}^{[\alpha]}_{q,0}\widetilde{I}^{[\alpha]-\alpha}_{q,0}x)(t),\ \alpha>0.\end{cases}$$

这里$[\alpha]$表示大于或等于α的最小整数.

性质1[16] 若$\alpha>0$, $a\leqslant b\leqslant t$,则$\widetilde{(t-a)}^{(\alpha)}\geqslant\widetilde{(t-b)}^{(\alpha)}$.

性质2[16] $\widetilde{[a(t-s)]}^{(\alpha)}=a^{\alpha}\widetilde{(t-s)}^{(\alpha)}$, ${}_tD_q\widetilde{(t-s)}^{(\alpha)}=\widetilde{[\alpha]}_q\widetilde{(q^{-1}t-s)}^{(\alpha-1)}$, ${}_sD_q\widetilde{(t-q^{-1}s)}^{(\alpha)}=-\widetilde{[\alpha]}_q\widetilde{(t-s)}^{(\alpha-1)}$.

引理1[16] $(\widetilde{D}_{q,0}\widetilde{I}_{q,0}x)(t)=x(t)$, $(\widetilde{I}_{q,0}\widetilde{D}_{q,0}x)(t)=x(t)-x(0)$.

引理2[16] 设$\alpha,\beta\geqslant0$, $x(t)$是定义在$[0,1]$上的连续函数,则有:

(a) $(\widetilde{I}^{\alpha}_{q,0}\widetilde{I}^{\beta}_{q,0}x)(t)=(\widetilde{I}^{\alpha+\beta}_{q,0}x)(t)$;

(b) $(\widetilde{D}^{\alpha}_{q,0}\widetilde{I}^{\alpha}_{q,0}x)(t)=x(t)$;

(c) 存在$c_i\in\mathbf{R}$, $i=1,2,\cdots,N$,使得$(\widetilde{I}^{\alpha}_{q,0}\widetilde{D}^{\alpha}_{q,0}x)(t)=x(t)+c_1t^{\alpha-1}+c_2t^{\alpha-2}+\cdots+c_Nt^{\alpha-N}(t\in[0,1]$,

$\alpha\in(N-1,N]$).

引理 3[16]　设 $\alpha\in\mathbf{R}^{+}/\mathbf{N}_0$, $\lambda\in(-1,+\infty)$,则:

(a) $\widetilde{I}_{q,0}^{\alpha}t^{\lambda}=\dfrac{\overline{\Gamma}_q(\lambda+1)}{\overline{\Gamma}_q(\lambda+\alpha+1)}q^{\binom{\alpha}{2}+\lambda\alpha}t^{\lambda+\alpha}$,

(b) $\widetilde{D}_{q,0}^{\alpha}t^{\lambda}=\dfrac{\overline{\Gamma}_q(\lambda+1)}{\overline{\Gamma}_q(\lambda-\alpha+1)}q^{\binom{\alpha}{2}-\lambda\alpha}t^{\lambda-\alpha}$.

引理 4[16]　设 $\alpha>0$, $x(t)$ 是定义在 $[0,1]$ 上的连续函数,则

$$(\widetilde{I}_{q,0}^{\alpha}\widetilde{D}_{q,0}x)(t)=(\widetilde{D}_{q,0}\widetilde{I}_{q,0}^{\alpha}x)(t)-\frac{t^{\alpha-1}}{\overline{\Gamma}_q(\alpha)}x(0).$$

引理 5　$\widetilde{D}_q\displaystyle\int_0^t f(t,s)\tilde{\mathrm{d}}_qs=\int_0^{q^{-1}t}\widetilde{D}_qf(t,s)\tilde{\mathrm{d}}_qs+f(qt,t)$.

证明　$\widetilde{D}_q\displaystyle\int_0^t f(t,s)\tilde{\mathrm{d}}_qs=\widetilde{D}_q\Big(t(1-q^2)\sum_{k=0}^{\infty}q^{2k}f(t,q^{2k+1}t)\Big)=\frac{1}{(q-q^{-1})t}\Big(qt(1-q^2)\sum_{k=0}^{\infty}q^{2k}f(qt,q^{2k+2}t)-q^{-1}t(1-q^2)\sum_{k=0}^{\infty}q^{2k}f(q^{-1}t,q^{2k}t)\Big)=\sum_{k=0}^{\infty}q^{2k+2}f(qt,q^{2k+2}t)+\sum_{k=0}^{\infty}q^{2k}f(q^{-1}t,q^{2k}t)=-\sum_{k=0}^{\infty}q^{2k}f(qt,q^{2k}t)+\sum_{k=0}^{\infty}q^{2k}f(q^{-1}t,q^{2k}t)+f(qt,t)=q^{-1}t(1-q^2)\sum_{k=0}^{\infty}q^{2k}\frac{f(qt,q^{2k}t)-f(q^{-1}t,q^{2k}t)}{(q-q^{-1})t}+f(qt,t)=\int_0^{q_t^{-1}}\widetilde{D}_qf(t,s)\tilde{\mathrm{d}}_qs+f(qt,t)$.

引理 6[16]　设 B 是 Banach 空间, $P\subseteq B$ 是一个锥, Ω_1 和 Ω_2 是包含于 B 的开子集,且 $0\in\Omega_1$, $\overline{\Omega}_1\subseteq\Omega_2$,并且进一步假设 $T:P\cap(\overline{\Omega}_2\backslash\Omega_1)\to\mathbf{R}$ 是全连续算子. T 在 $P\cap(\overline{\Omega}_2\backslash\Omega_1)$ 内至少有一个不动点,如果下列条件之一成立:

1) $\|Ty\|\leqslant\|y\|$, $y\in P\cap\partial\Omega_1$, $\|Ty\|\geqslant\|y\|$, $y\in P\cap\partial\Omega_2$;

2) $\|Ty\|\geqslant\|y\|$, $y\in P\cap\partial\Omega_1$, $\|Ty\|\leqslant\|y\|$, $y\in P\cap\partial\Omega_2$.

2　主要结果及其证明

考虑非线性分数阶 q-对称差分方程边值问题:

$$(\widetilde{D}_{q,0}^{\alpha}x)(t)=-f(q^{-\alpha}t,x(q^{-\alpha}t)),\ 0<t<q^{\alpha};\tag{1}$$

$$x(0)=(\widetilde{D}_qx)(0)=0,\ (\widetilde{D}_qx)(1)=\beta\geqslant0,\tag{2}$$

其中 $2<\alpha\leqslant3$,且 $f:[0,1]\times\mathbf{R}\to\mathbf{R}$ 是非负连续函数. 利用引理 2 知,

$$x(t)=-\widetilde{I}_{q,0}^{\alpha}f(q^{-\alpha}t,x(q^{-\alpha}t))+c_1t^{\alpha-1}+c_2t^{\alpha-2}+c_3t^{\alpha-3},\ 0<t<1.\tag{3}$$

由条件 $x(0)=0$ 得 $c_3=0$. 对式(3)两边求导并利用引理 5 得

$$\widetilde{D}_qx(t)=-\widetilde{D}_q\widetilde{I}_{q,0}^{\alpha}f(q^{-\alpha}t,x(q^{-\alpha}t))+c_1\,[\overline{\alpha-1}]_q\,t^{\alpha-2}+c_2\,[\overline{\alpha-2}]_q\,t^{\alpha-2}=$$
$$-\frac{q^{\binom{\alpha}{2}}}{\overline{\Gamma}_q(\alpha)}\int_0^{q^{-1}t}\widetilde{D}_q\,\overline{(t-s)}^{(\alpha-1)}f(q^{-1}s,x(q^{-1}s))\tilde{\mathrm{d}}_qs+c_1\,[\overline{\alpha-1}]_q\,t^{\alpha-2}+c_2\,[\overline{\alpha-2}]_q\,t^{\alpha-2}=$$
$$-\frac{q^{\binom{\alpha}{2}}}{\overline{\Gamma}_q(\alpha)}\int_0^{q^{-1}t}[\overline{\alpha-1}]_q\,\overline{(q^{-1}t-s)}^{(\alpha-1)}f(q^{-1}s,x(q^{-1}s))\tilde{\mathrm{d}}_qs+c_1\,[\overline{\alpha-1}]_q\,t^{\alpha-2}+c_2\,[\overline{\alpha-2}]_q\,t^{\alpha-2}.$$

将边界条件 $(\widetilde{D}_qx)(0)=0$ 代入上式有 $c_2=0$,再利用条件 $(\widetilde{D}_qx)(1)=\beta\geqslant0$,有

$$\widetilde{D}_qx(1)=-\frac{q^{\binom{\alpha}{2}}}{\overline{\Gamma}_q(\alpha-1)}\int_0^{q^{-1}}\overline{(q^{-1}-s)}^{(\alpha-2)}f(q^{-1}s,x(q^{-1}s))\tilde{\mathrm{d}}_qs+c_1\,[\overline{\alpha-1}]_q=\beta,$$

所以 $c_1=\dfrac{q^{\binom{\alpha}{2}-\alpha+2}}{\overline{\Gamma}_q(\alpha)}\displaystyle\int_0^{q^{-1}}\overline{(q^{-1}-s)}^{(\alpha-2)}f(q^{-1}s,x(q^{-1}s))\tilde{\mathrm{d}}_qs+\frac{\beta}{[\overline{\alpha-1}]_q}$. 由以上结果知

$$x(t)=\left(\frac{q^{\binom{\alpha}{2}}}{\widetilde{\Gamma}_q(\alpha)}\int_0^{q^{-1}}\overline{(q^{-1}-s)}^{(\alpha-2)}f(q^{-1}s,x(q^{-1}s))\tilde{\mathrm{d}}_qs+\frac{\beta}{[\alpha-1]_q q^{\alpha-2}}\right)t^{\alpha-1}-$$

$$\frac{q^{\binom{\alpha}{2}}}{\widetilde{\Gamma}_q(\alpha)}\int_0^t\overline{(t-s)}^{(\alpha-1)}f(q^{-1}s,x(q^{-1}s))\tilde{\mathrm{d}}_qs=\int_0^{q^{-1}}G(t,s)f(q^{-1}s,x(q^{-1}s))\tilde{\mathrm{d}}_qs+h_f(t),$$

其中 $G(t,s)=\frac{1}{\widetilde{\Gamma}_q(\alpha)}q^{\binom{\alpha}{2}}\begin{cases}\overline{(q^{-1}-s)}^{(\alpha-2)}t^{\alpha-1}-\overline{(t-s)}^{(\alpha-1)},\ 0\leqslant s\leqslant q^{-1}t\leqslant q^{-1};\\ \overline{(q^{-1}-s)}^{(\alpha-2)}t^{\alpha-1},\ 0\leqslant q^{-1}t\leqslant s\leqslant q^{-1},\end{cases}$ 且

$$h_f(t)=\frac{\beta}{[\alpha-1]_q}t^{\alpha-1}+\frac{1}{\widetilde{\Gamma}_q(\alpha)}q^{\binom{\alpha}{2}}\int_t^{q^{-1}t}\overline{(t-s)}^{(\alpha-1)}f(q^{-1}s,q^{-1}s)\tilde{\mathrm{d}}_qs,\ t\in[0,1].$$

引理 7 函数 $G(t,s)$ 有如下性质:

$$G(t,s)\geqslant 0,\ G(t,s)\leqslant G(1,s),\ (t,s)\in[0,1]\times[0,q^{-1}];\tag{4}$$

$$G(t,s)\geqslant g(t)G(1,s),\ (t,s)\in[0,1]\times[0,q^{-1}],\ g(t)=t^{\alpha-1}.\tag{5}$$

证明 记 $g_1(t,s)=\overline{(q^{-1}-s)}^{(\alpha-2)}t^{\alpha-1}-\overline{(t-s)}^{(\alpha-1)}$, $g_2(t,s)=\overline{(q^{-1}-s)}^{(\alpha-2)}t^{\alpha-1}$,显然 $g_2(t,s)\geqslant 0$. 对 $t\neq 0$,有 $g_1(t,s)=\overline{(q^{-1}-s)}^{(\alpha-2)}t^{\alpha-1}-\overline{(t-s)}^{(\alpha-1)}=t^{\alpha-1}\left(\overline{(q^{-1}-s)}^{(\alpha-2)}-\overline{(1-\frac{s}{t})}^{(\alpha-1)}\right)\geqslant t^{\alpha-1}\left(\overline{(1-s)}^{(\alpha-2)}-\overline{(1-s)}^{(\alpha-1)}\right)\geqslant 0$,所以 $G(t,s)\geqslant 0$. 进一步,对 $t\in[0,1]$,有 $\widetilde{D}_qg_1(t,s)=\overline{[\alpha-1]}_q\cdot\overline{(q^{-1}-s)}^{(\alpha-2)}t^{\alpha-2}-\overline{[\alpha-1]}_q\overline{(q^{-1}t-s)}^{(\alpha-2)}\geqslant\overline{[\alpha-1]}_qt^{\alpha-2}\left(\overline{(1-s)}^{(\alpha-2)}-\overline{(1-\frac{s}{t})}^{(\alpha-2)}\right)\geqslant 0$,即 $g_1(t,s)$ 关于 t 是递增的. 又由于 $g_2(t,s)$ 关于 t 显然是递增的,由此知 $G(t,s)$ 关于 t 是递增的,并且有 $G(t,s)\leqslant G(1,s)$, $0\leqslant t\leqslant 1$.

假设 $q^{-1}t\geqslant s$,则 $\frac{G(t,s)}{G(1,s)}=\frac{\overline{(q^{-1}-s)}^{(\alpha-2)}t^{\alpha-1}-\overline{(t-s)}^{(\alpha-1)}}{\overline{(q^{-1}-s)}^{(\alpha-2)}-\overline{(1-s)}^{(\alpha-1)}}=t^{\alpha-1}\frac{\overline{(1-s)}^{(\alpha-2)}-\overline{(1-\frac{s}{t})}^{(\alpha-1)}}{\overline{(1-s)}^{(\alpha-2)}-\overline{(1-s)}^{(\alpha-1)}}\geqslant t^{\alpha-1}$;假设 $q^{-1}t\leqslant s$,则 $\frac{G(t,s)}{G(1,s)}=\frac{\overline{(q^{-1}-s)}^{(\alpha-2)}t^{\alpha-1}}{\overline{(q^{-1}-s)}^{(\alpha-2)}-\overline{(1-s)}^{(\alpha-1)}}=t^{\alpha-1}\frac{\overline{(q^{-1}-s)}^{(\alpha-2)}}{\overline{(q^{-1}-s)}^{(\alpha-2)}-\overline{(1-s)}^{(\alpha-1)}}\geqslant t^{\alpha-1}$. 引理 7 得证.

注 1 根据引理 7 有,

$$\min_{t\in[\tau,1]}G(t,s)\geqslant\tau^{\alpha-1}G(1,s),\ s\in[0,q^{-1}],\ \tau\in(0,1).\tag{6}$$

引理 8 函数 $h_f(t)\geqslant 0$,且单调递增.

证明 由 $\overline{(q^{-1}t-s)}^{(\alpha-2)}=(q^{-1}t)^{\alpha-2}\frac{\prod_{i=0}^{\infty}(1-\frac{s}{q^{-1}t}q^{2i+1})}{\prod_{i=0}^{\infty}(1-\frac{s}{q^{-1}t}q^{2(i+\alpha-1)+1})}\geqslant 0$ 及函数 f 的非负连续性,有 $\widetilde{D}_qh_f(t)=\beta t^{\alpha-2}+\frac{1}{\widetilde{\Gamma}_q(\alpha-1)}q^{\binom{\alpha}{2}}\int_{q^{-1}t}^{q^{-2}t}\overline{(q^{-1}t-s)}^{(\alpha-2)}f(q^{-1}s,q^{-1}s)\tilde{\mathrm{d}}_qs\geqslant 0$, $t\in[0,1]$. 又因 $h_f(0)=0$,所以函数 $h_f(t)\geqslant 0$.

设 $B=C[0,1]$ 是一个具有上确界模 $\|x\|=\sup_{t\in[0,1]}|x(t)|$ 的 Banach 空间,令 $\tau=q^n$, $n\in\mathbf{N}$,并且定义一个锥 $P=\{x\in B,x(t)\geqslant 0,\ \min_{t\in[\tau,1]}x(t)\geqslant\tau^{\alpha-1}\|x\|\}$.

注 2 定义算子 $T:P\to B$, $(Tx)(t)=\int_0^{q^{-1}}G(t,s)f(q^{-1}s,x(q^{-1}s))\tilde{\mathrm{d}}_qs+h_f(t)$,则 $T(P)\subset P$,且 T 是全连续.

为了给出本文主要结果,定义如下两个常数:

$$M=\int_0^{q^{-1}}G(1,s)\tilde{\mathrm{d}}_qs+\frac{1}{\widetilde{\Gamma}_q(\alpha-1)}q^{\binom{\alpha}{2}}\int_1^{q^{-1}}\overline{(1-s)}^{(\alpha-1)}\tilde{\mathrm{d}}_qs,\ N=\max_{t\in[0,1]}\int_\tau^{q^{-1}}G(t,s)\tilde{\mathrm{d}}_qs.$$

定理 1 令 $\tau=q^n$, $n\in\mathbf{N}$. 假设函数 $f(t,x)$ 是定义在$[0,1]\times[0,\infty)$上的非负连续函数. 如果存在

两个正常数 $r_2 \geqslant r_1 > 0$ 使得

$$\frac{\beta}{[\alpha-1]_q} + M \max_{(t,x)\in[0,1]\times[0,r_1]} f(t,x) \leqslant r_1, \tag{7}$$

$$\frac{\beta}{[\alpha-1]_q} + N \min_{(t,x)\in[\tau,1]\times[\tau^{\alpha-1}r_2,r_2]} f(t,x) \geqslant r_2, \tag{8}$$

那么问题(1) 和(2) 有一个解满足 $x(t) > 0$, $t \in (0,1]$.

证明　令 $\Omega_1 = \{x \in P, \| x \| < r_1\}$. 对 $x \in P \cap \partial\Omega_1$, 有 $0 \leqslant x(t) \leqslant r_1$, $t \in [0,1]$. 利用公式(4) 和(7) 可得

$$\| Tx \| = \max_{t\in[0,1]}\left[\int_0^{q^{-1}} G(t,s) f(q^{-1}s, x(q^{-1}s))\tilde{\mathrm{d}}_q s + h_f(t)\right] = \int_0^{q^{-1}} G(1,s) f(q^{-1}s, x(q^{-1}s))\tilde{\mathrm{d}}_q s +$$

$$\frac{\beta}{[\alpha-1]_q} + \int_1^{q^{-1}} \overline{(1-s)}^{(\alpha-1)} f(q^{-1}s, x(q^{-1}s))\tilde{\mathrm{d}}_q s \leqslant M \max_{(t,x)\in[0,1]\times[0,r_1]} f(t,x) + \frac{\beta}{[\alpha-1]_q} \leqslant r_1 = \| x \|.$$

令 $\Omega_2 = \{x \in P, \| x \| < r_2\}$. 对 $x \in P \cap \partial\Omega_2$, 有 $\tau^{\alpha-1} r_2 \leqslant x(t) \leqslant r_2$, $t \in [\tau,1]$. 利用公式(6) 和(8) 可得

$$\| Tx \| = \max_{t\in[0,1]}\left[\int_0^{q^{-1}} G(t,s) f(q^{-1}s, x(q^{-1}s))\tilde{\mathrm{d}}_q s + h_f(t)\right] = \int_0^{q^{-1}} G(1,s) f(q^{-1}s, x(q^{-1}s))\tilde{\mathrm{d}}_q s +$$

$$\frac{\beta q^{2-\alpha}}{[\alpha-1]_q} \geqslant N \min_{(t,x)\in[\tau,1]\times[\tau^{\alpha-1}r_2,r_2]} f(t,x) + \frac{\beta q^{2-\alpha}}{[\alpha-1]_q} \geqslant r_2 = \| x \|.$$

由引理 6 知, T 在 P 上存在一个不动点使得 $r_1 \leqslant \| x \| \leqslant r_2$ 且

$$x = \int_0^{q^{-1}} G(t,s) f(q^{-1}s, x(q^{-1}s))\tilde{\mathrm{d}}_q s + \frac{\beta q^{2-\alpha} t^{\alpha-1}}{[\alpha-1]_q} \geqslant$$

$$t^{\alpha-1}\left[\int_0^{q^{-1}} G(1,s) f(q^{-1}s, x(q^{-1}s))\tilde{\mathrm{d}}_q s + \frac{\beta q^{2-\alpha}}{[\alpha-1]_q}\right] = t^{\alpha-1} \| x \|,$$

所以 $x(t) > 0$, $t \in (0,1]$. 证毕.

例 1　在问题(1) 和(2) 中,取 $\alpha = 2.5$, $q = \tau = 0.5$, $\beta = 0$, $f(t,x) = x^2(2 + \sin x)$. 利用函数 $G(t,s)$ 的定义,通过计算可得

$$M = \int_0^{q^{-1}} G(1,s)\tilde{\mathrm{d}}_q s + \frac{1}{\tilde{\Gamma}(\alpha-1)} q^{\binom{\alpha}{2}} \int_1^{q^{-1}} \overline{(1-s)}^{(\alpha-1)} \tilde{\mathrm{d}}_q s =$$

$$\left(\frac{-1}{[\alpha-1]_q} \overline{(q^{-1}-q^{-1}s)}^{(\alpha-1)} + \frac{1}{[\alpha]_q} \overline{(1-q^{-1}s)}^{(\alpha)}\right)\Bigg|_0^{q^{-1}} + \frac{1}{\tilde{\Gamma}_q(\alpha)} q^{\binom{\alpha}{2}} \overline{(1-q^{-1}s)}^{(\alpha)}\Big|_0^{q^{-1}} \leqslant$$

$$\frac{1}{[\alpha-1]_q} q^{1-\alpha} + \frac{1}{[\alpha]_q} \overline{(1-q^{-2})}^{(\alpha)} - \frac{1}{[\alpha]_q} \leqslant \frac{1}{[\alpha-1]_q} q^{1-\alpha} < 2.5,$$

则 $M \max_{(t,x)\in[0,1]\times[0,r_1]} f(t,x) \leqslant 2.5 \times 3r_1^2$. 选择 $r_1 \leqslant \frac{1}{7.5}$, 则式(7) 成立.

同理,

$$N = \max_{t\in[0,1]} \int_\tau^{q^{-1}} G(t,s)\tilde{\mathrm{d}}_q s \geqslant \int_{0.5}^{q^{-1}} G(1,s)\tilde{\mathrm{d}}_q s = \left(\frac{-1}{[\alpha-1]_q} \overline{(q^{-1}-q^{-1}s)}^{(\alpha-1)} +\right.$$

$$\left.\frac{1}{[\alpha]_q} \overline{(1-q^{-1}s)}^{(\alpha)}\right)\Bigg|_{0.5}^{q^{-1}} = \frac{1}{[\alpha-1]_q} \overline{(q^{-1}-1)}^{(\alpha-1)} + \frac{1}{[\alpha]_q} \overline{(1-q^{-2})}^{(\alpha)} - \frac{1}{[\alpha]_q} \overline{(1-1)}^{(\alpha)} =$$

$$\frac{q^{1-\alpha}}{[\alpha-1]_q} \prod_{i=0}^{\infty} \frac{(1-q^{2i+2})}{(1-q^{2(i+\alpha)})} + \frac{1}{[\alpha]_q} \prod_{i=0}^{\infty} \frac{(1-q^{2i-1})}{(1-q^{2(i+\alpha)-1})} - \frac{1}{[\alpha]_q} \prod_{i=0}^{\infty} \frac{(1-q^{2i+1})}{(1-q^{2(i+\alpha)+1})} =$$

$$\frac{q^{1-\alpha}}{[\alpha-1]_q} \prod_{i=0}^{\infty} \frac{(1-q^{2i+2})}{(1-q^{2(i+\alpha)})} - q^{-1}(1-q^2) \prod_{i=0}^{\infty} \frac{(1-q^{2i+1})}{(1-q^{2i+4})} =$$

$$q^{-1}(1-q^2)\left[\frac{8\sqrt{2}}{7}\prod_{i=0}^{\infty}\frac{(1-q^{2i+2})}{(1-q^{2(i+\alpha)})}-\prod_{i=0}^{\infty}\frac{(1-q^{2i+1})}{(1-q^{2i+4})}\right]\geqslant$$

$$q^{-1}(1-q^2)\left[\frac{8\sqrt{2}}{7}-1\right]\prod_{i=0}^{\infty}\frac{(1-q^{2i+2})}{(1-q^{2(i+\alpha)})}\geqslant 0.6.$$

选择 $r_2\geqslant\frac{5}{3}$,则式(8)成立.由定理 1 知边值问题(1)和(2)有一个解满足 $x(t)>0$, $t\in(0,1]$.

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