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# 带有分数阶边界条件的一类 Caputo 差分方程边值问题

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**摘要:** 运用分数阶的基本定义和引理, 讨论了一类带有分数阶边界条件的离散 Caputo 分数阶差分方程边值问题的格林函数, 并给出了  $\nu = 2$  时格林函数的几个重要性质.  
**关键词:** 离散型; Caputo 边值问题; 分数阶边值条件; 格林函数  
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## On a Caputo fractional boundary value problem for fractional boundary conditions

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**Abstract:** Green's function of a discrete Caputo fractional boundary value problem for fractional boundary conditions is discussed by applying some fractional basic definitions and lemmas. Meanwhile, we give some important properties of Green's function in  $\nu = 2$ .  
**Key words:** discrete; Caputo fractional boundary condition; boundary value problem; Green's function

分数阶微分方程在电气工程、化学、生物数学和控制理论等众多领域有着广泛的应用<sup>[1-4]</sup>. 近年来, 分数阶差分方程理论和在任意时间尺度上的分数阶微积分一般理论都得到了相应的发展. Goodrich<sup>[5]</sup> 对一个离散的分数阶初始值边值问题给出了一些连续性的结果; Atici 和 Elloe<sup>[6-8]</sup> 及 Goodrich<sup>[9-10]</sup> 对分数阶差分方程边值问题进行了研究, 推导了其格林函数, 得到了一些解的存在性结果. 在经典的离散问题中(即  $\mu=2$ ), 多点问题也得到了研究<sup>[11-13]</sup>, 但对于离散的 Caputo 分数阶方程边值问题的研究还比较少. 本文将讨论下面的 Caputo 分数阶差分方程边值问题(FBVP):

$$\begin{cases} -\Delta_c^\alpha y(t) = f(t+\nu-1, y(t+\nu-1)), & t \in [0, b+1]_{\mathbb{N}_0}; \\ y(\nu-2) = [\Delta^\alpha y(t)]_{t=\nu+b-\alpha+1} = 0. \end{cases} \quad (1)$$

其中  $1 < \nu \leq 2$ ,  $0 \leq \alpha < 1$ ,  $f: [0, 1] \times \mathbf{R} \rightarrow \mathbf{R}$  是连续函数. 易见, 当  $\alpha=0$  时, 边值条件为  $y(\nu+b+1)=0$ ; 当  $\alpha=1$  时, 边值条件为  $\Delta y(\nu+b)=0$ . 因此, 问题(1) 可以看作是共轭的 Caputo 分数阶边值问题<sup>[14-15]</sup>.

## 1 预备知识

首先给出一些必要的分数阶定义和运算结果.

**定义 1** 在  $\mathbf{N}_a := \{a, a+1, \dots\}$  的函数的  $\nu$  阶和分定义为  $\Delta^{-\nu} f(t) := \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\underline{\nu-1}} f(s)$ ,

其中  $\nu > 0$ ,  $t \in \{a+\nu, a+\nu+1, \dots\} := \mathbf{N}_{a+\nu}$ . 同时称函数的  $\nu$  阶差分为  $\Delta_C^\nu f(t) = \Delta^{-(N-\nu)} \Delta^N f(t)$ , 其中  $\nu > 0$ ,  $0 \leq N-1 < \nu \leq N$ ,  $N \in \mathbf{N}$ ,  $t \in \mathbf{N}_{a+N-\nu}$ .

**定义 2** 对任意的  $t$  和  $\nu$ ,  $t^\nu = \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$  称为右置的,  $\nu$  是伽马函数的一个极点且  $t+1$  不是极点,

那么  $t^\nu = 0$ .

**引理 1** 若  $t$  和  $\nu$  是任意数,  $t^\nu$  和  $t^{\nu-1}$  有定义, 那么  $\Delta t^\nu = \nu t^{\nu-1}$ .

**引理 2** 若  $f$  是实值函数, 且  $\mu, \nu > 0$ , 则对任意的  $t$ , 如  $t = a + \mu + \nu$ , 有  $\Delta^{-\nu} [\Delta^{-\mu} f(t)] = \Delta^{-(\nu+\mu)} f(t) = \Delta^{-\mu} [\Delta^{-\nu} f(t)]$ , 其中  $t \in \mathbf{N}_{a+\nu+\mu}$ .

**引理 3** 若  $0 \leq N-1 < \nu \leq N$ , 那么  $\Delta_{a+(n-\nu)}^{-\nu} \Delta_C^\nu f(t) = f(t) - \sum_{k=0}^{N-1} C_k (t-a)^k$ ,  $C_i \in \mathbf{R}$  且  $1 \leq i \leq N$ .

**引理 4**<sup>[15]</sup> 对  $\beta > 0$  和所有的  $\mu \in \mathbf{R}$ , 有  $\Delta^\beta t^\mu = \frac{\Gamma(\mu+1)t^{\mu-\beta}}{\Gamma(\mu-\beta+1)}$ .

## 2 多点边值条件和格林函数的推导

**定理 1**<sup>[15]</sup> 假设边值条件  $y(\nu-2) = [\Delta^a y(t)]_{t=\nu+b-a+1} = 0$  成立, 那么存在一个函数  $\psi: \mathbf{R}^{b+4} \rightarrow \mathbf{R}$  使得  $y(\nu+b+1) = \psi(y)$ , 其中向量  $\mathbf{y} := (y(\nu-2), y(\nu-1), y(\nu), \dots, y(\nu+b+1))$ .

**定理 2** 令  $h: [\nu-1, \nu+b]_{\mathbf{N}_{\nu-1}} \rightarrow \mathbf{R}$ , 那么方程(1)的唯一解为  $y(t) := \sum_{s=0}^{b+1} G(t, s) h(s+\nu-1)$ , 其中  $G(t, s)$  是边值问题(2)的格林函数:

$$G(t, s) = \begin{cases} \frac{(2+t-\nu)\Gamma(2-\alpha)}{\Gamma(\nu-\alpha)(\nu+b-\alpha+1)^{1-\alpha}} (\nu+b-\alpha-s)^{\underline{\nu-\alpha-1}} - \frac{(t-s-1)^{\underline{\nu-1}}}{\Gamma(\nu)}, & 0 \leq s < t-\nu+1 \leq b+1; \\ \frac{(2+t-\nu)\Gamma(2-\alpha)}{\Gamma(\nu-\alpha)(\nu+b-\alpha+1)^{1-\alpha}} (\nu+b-\alpha-s)^{\underline{\nu-\alpha-1}}, & 0 \leq t-\nu+1 < s \leq b+1, \end{cases}$$

$$\begin{cases} -\Delta_C^\nu y(t) = h(t+\nu-1), & t \in [0, b+1]_{\mathbf{N}_0}; \\ y(\nu-2) = [\Delta^a y(t)]_{t=\nu+b-a+1} = 0, \end{cases} \quad (2)$$

其中  $1 < \nu \leq 2$ ,  $0 \leq \alpha < 1$ .

**证明** 由引理 3 可以推出方程(2)的一个等价方程为

$$y(t) = -\frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t-s-1)^{\underline{\nu-1}} h(s+\nu-1) + C_1 + C_2 t,$$

其中  $C_i \in \mathbf{R}$ ,  $i=1, 2$ . 由  $y(\nu-2) = 0$  可知,  $C_1 = -C_2(\nu-2)$ . 另一方面, 边值条件  $[\Delta^a y(t)]_{t=\nu+b-a+1} = 0$  隐含了

$$0 = [\Delta^a y(t)]_{t=\nu+b-a+1} = [-\Delta^{a-\nu} h(t)]_{t=\nu+b-a+1} + [\Delta^a C_1]_{t=\nu+b-a+1} + [\Delta^a C_2 t]_{t=\nu+b-a+1}. \quad (3)$$

由引理 4 可知

$$[\Delta^a C_1]_{t=\nu+b-a+1} = 0, [\Delta^a C_2 t]_{t=\nu+b-a+1} = C_2 \left[ \frac{\Gamma(2)t^{1-\alpha}}{\Gamma(2-\alpha)} \right]_{t=\nu+b-a+1} = C_2 \frac{(\nu+b-\alpha+1)^{1-\alpha}}{\Gamma(2-\alpha)}. \quad (4)$$

于是有  $[-\Delta^{a-\nu} h(t)]_{t=\nu+b-a+1} = [-\frac{1}{\Gamma(\nu-\alpha)} \sum_{s=0}^{t+\alpha-\nu} (t-s-1)^{\underline{\nu-\alpha-1}} h(s+\nu-1)]_{t=\nu+b-a+1} = -\frac{1}{\Gamma(\nu-\alpha)} \sum_{s=0}^{b+1} (\nu+b-\alpha-s)^{\underline{\nu-\alpha-1}} h(s+\nu-1)$ . 由(3)和(4)式计算得:

$$C_2 = \frac{\Gamma(2-\alpha)}{\Gamma(\nu-\alpha)(\nu+b-\alpha+1)^{1-\alpha}} \sum_{s=0}^{b+1} (\nu+b-\alpha-s)^{\underline{\nu-\alpha-1}} h(s+\nu-1),$$

$$C_1 = -\frac{(\nu-2)\Gamma(2-\alpha)}{\Gamma(\nu-\alpha)(\nu+b-\alpha+1)^{1-\alpha}} \sum_{s=0}^{b+1} (\nu+b-\alpha-s)^{\underline{\nu-\alpha-1}} h(s+\nu-1).$$

由以上结果可以推出  $y(t)$  的表达式为

$$y(t) = -\frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t-s-1)^{\underline{\nu-1}} h(s+\nu-1) - \frac{(\nu-2)\Gamma(2-\alpha)}{\Gamma(\nu-\alpha)(\nu+b-\alpha+1)^{1-\alpha}} \sum_{s=0}^{b+1} (\nu+b-\alpha-s)^{\underline{\nu-\alpha-1}} h(s+\nu-1) + \frac{\Gamma(2-\alpha)t}{\Gamma(\nu-\alpha)(\nu+b-\alpha+1)^{1-\alpha}} \sum_{s=0}^{b+1} (\nu+b-\alpha-s)^{\underline{\nu-\alpha-1}} h(s+\nu-1) = -\frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t-s-1)^{\underline{\nu-1}} h(s+\nu-1) + \frac{(2+t-\nu)\Gamma(2-\alpha)}{\Gamma(\nu-\alpha)(\nu+b-\alpha+1)^{1-\alpha}} \sum_{s=0}^{b+1} (\nu+b-\alpha-s)^{\underline{\nu-\alpha-1}} h(s+\nu-1).$$

### 3 格林函数的性质

**定理 3**  $G(t, s)$  是定理 2 中的格林函数, 当  $\nu=2$  时, 有:

- ① 对  $(t, s) \in [1, 2+b]_{\mathbb{N}_{\nu-1}} \times [0, b]_{\mathbb{N}_0}$ ,  $G(t, s) > 0$ ;
- ②  $\max_{t \in [1, 1+b]_{\mathbb{N}_{\nu-1}}} G(t, s) = G(s+1, s)$ ,  $s \in [0, b]_{\mathbb{N}_0}$ ;
- ③ 存在实数  $\gamma \in (0, 1)$  使得对  $s \in [0, b]_{\mathbb{N}_0}$ ,  $\min_{t \in [\frac{b+2}{4}, \frac{3(b+2)}{4}]} G(t, s) \geq \gamma \cdot \max_{t \in [1, 2+b]_{\mathbb{N}_1}} G(t, s) = \gamma G(s+1, s)$ .

**证明** 1) 对于  $0 \leq t-1 < s \leq b+1$ ,  $G(t, s) > 0$  显然成立, 现只需证明  $0 \leq s < t-1 \leq b+1$  的情况. 当  $0 \leq s < t-1 \leq b+1$  时,

$$G(t, s) = \frac{t}{(2+b-\alpha+1)^{1-\alpha}} (2+b-\alpha-s)^{1-\alpha} - \frac{(t-s-1)}{\Gamma(2)} = t \left[ \frac{(2+b-\alpha-s)^{1-\alpha}}{(2+b-\alpha+1)^{1-\alpha}} - 1 \right] + s+1 > (s+1) \left[ \frac{(2+b-\alpha-s)^{1-\alpha}}{(2+b-\alpha+1)^{1-\alpha}} - 1 \right] + s+1 = (s+1) \frac{(2+b-\alpha-s)^{1-\alpha}}{(2+b-\alpha+1)^{1-\alpha}} > 0.$$

2) 当  $0 \leq s < t-1 \leq b+1$  时,  $\Delta_t G(t, s) = \frac{(2+b-\alpha-s)^{1-\alpha}}{(2+b-\alpha+1)^{1-\alpha}} - 1$ . 因为  $(2+b-\alpha-s)^{1-\alpha} < (2+b-\alpha+1)^{1-\alpha}$ , 所以  $\Delta_t G(t, s) < 0$ . 当  $0 \leq t-1 < s \leq b+1$  时,  $\Delta_t G(t, s) = \frac{(2+b-\alpha-s)^{1-\alpha}}{(2+b-\alpha+1)^{1-\alpha}} > 0$ . 当  $0 \leq s < t-1 \leq b+1$  时,  $G(t, s)$  关于  $t$  是递减的; 当  $0 \leq t-1 < s \leq b+1$  时,  $G(t, s)$  关于  $t$  是递增的. 因此可以得到  $\max_{t \in [1, 1+b]_{\mathbb{N}_{\nu-1}}} G(t, s) = G(s+1, s)$ ,  $s \in [0, b]_{\mathbb{N}_0}$ .

3) 因为

$$\frac{G(t, s)}{G(s+1, s)} = \begin{cases} \frac{t}{s+1} - \frac{(t-s-1)(2+b-\alpha+1)^{1-\alpha}}{(s+1)(2+b-\alpha-s)^{1-\alpha}}, & 0 \leq s < t-1 \leq b+1; \\ \frac{t}{s+1}, & 0 \leq t-1 < s \leq b+1. \end{cases}$$

所以当  $t-1 < s$  和  $t \in [\frac{b+2}{4}, \frac{3(b+2)}{4}]$  时, 有  $\frac{G(t, s)}{G(s+1, s)} \geq \frac{1}{b+2}$ ; 当  $s < t-1$  和  $t \in [\frac{b+2}{4}, \frac{3(b+2)}{4}]$  时,

$$\text{有 } \min_{t \in [\frac{b+2}{4}, \frac{3(b+2)}{4}]} \frac{G(t, s)}{G(s+1, s)} = \frac{\frac{3(b+2)}{4}}{s+1} - \frac{(\frac{3(b+2)}{4} - s - 1)(2+b-\alpha+1)^{1-\alpha}}{(s+1)(2+b-\alpha-s)^{1-\alpha}}.$$

下面证明

$$\frac{\frac{3(b+2)}{4}}{s+1} - \frac{(\frac{3(b+2)}{4} - s - 1)(2+b-\alpha+1)^{1-\alpha}}{(s+1)(2+b-\alpha-s)^{1-\alpha}} \geq \frac{1}{b+2},$$

即  $\frac{3(b+2)}{s+1} - \frac{1}{b+2} \geq \frac{(\frac{3(b+2)}{4} - s - 1)(2+b-\alpha+1)^{\frac{1-\alpha}{4}}}{(s+1)(2+b-\alpha-s)^{\frac{1-\alpha}{4}}}$ . 令  $\frac{(2+b-\alpha+1)^{\frac{1-\alpha}{4}}}{(2+b-\alpha-s)^{\frac{1-\alpha}{4}}} = \mu$ , 则上式可化为

$$\frac{3(b+2)}{s+1}(1-\mu) + \mu - \frac{1}{b+2} \geq 0.$$

事实上, 因为  $s < t-1 \leq \frac{3(b+2)}{4} - 1 = \frac{3b+2}{4}$ , 所以  $s+1 < \frac{3(b+2)}{4}$ ,

$$\text{所以 } \frac{3(b+2)}{s+1}(1-\mu) + \mu - \frac{1}{b+2} > 1-\mu + \mu - \frac{1}{b+2} > 0 \text{ 成立.}$$

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