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具有强迫项的二阶时滞微分方程的振动性

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摘要: 讨论具有强迫项的二阶时滞微分方程 $(k(t)\varphi(x(t))x'(t))' + p(t)x(\tau(t)) + q(t)x(\sigma(t)) = e(t)$, 利用其线性近似方程 $(k(t)x'(t))' + p(t)x(\tau(t)) + q(t)x(\sigma(t)) = e(t)$ 的振动性, 给出了方程解振动的一个充分条件, 所得结果推广了文献[4]的相关结果.

关键词: 时滞微分方程; 强迫项; 振动性

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Oscillation of second order nonlinear delay differential equations with forced term

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Abstract: We consider the second order force differential equation with oscillatory potentials $(k(t)\varphi(x(t))x'(t))' + p(t)x(\tau(t)) + q(t)x(\sigma(t)) = e(t)$. A sufficient condition for oscillation of the equation is obtained by using the oscillation of its linearized equation $(k(t)x'(t))' + p(t)x(\tau(t)) + q(t)x(\sigma(t)) = e(t)$, which generalized the corresponding result of reference [4].

Key words: delay differential equation; forced; oscillation

振动理论是微分方程定性理论的一个重要分支, 其中时滞微分方程的振动理论受到国内外学者的广泛关注^[1-4]. 文献[2]考虑了方程 $(k(t)x'(t))' + p(t)|x(\tau(t))|^{\alpha-1}x(\tau(t)) = e(t)$ 的振动性, 其中 $\alpha \geq 1$, $p(t)$ 和 $e(t)$ 是可变号的. 文献[3]考虑了方程 $x''(t) + q(t)f(x(\tau(t))) = e(t)$ 的振动性, 其中 $f(x)$ 满足一定的增长条件. 文献[4]考虑了带有阻尼项的二阶微分方程 $(k(t)x'(t))' + p(t)|x(\tau(t))|^{\alpha-1} \cdot x(\tau(t)) + q(t)|x(\sigma(t))|^{\beta-1}x(\sigma(t)) = e(t)$ 的振动性, 其中 $\alpha \geq 1$, $\beta \geq 1$, $e(t)$ 是可变号的.

本文考虑具有强迫项的二阶时滞微分方程 $(k(t)\varphi(x(t))x'(t))' + p(t)x(\tau(t)) + q(t)x(\sigma(t)) = e(t)$, 得到了方程解振动的充分条件. 当 $\varphi(t) = 0$ 时, 本文结果推广了文献[4]中的相关结论.

1 主要结果及其证明

考虑具有强迫项的二阶微分方程

$$(k(t)\varphi(x(t))x'(t))' + p(t)x(\tau(t)) + q(t)x(\sigma(t)) = e(t), \tag{E_1}$$

其中: $k \in C((0, \infty), (0, \infty))$; $p, q, e \in C((0, \infty), \mathbf{R})$; $\tau \in C((0, \infty), (0, \infty))$ 且 $\tau(t) \leq t$, $\sigma \in C((0, \infty), (0, \infty))$ 且 $\sigma(t) \geq t$. 为方便起见, 先给出如下两个条件:

(A₁) 存在正常数 $k_2 \geq k_1$, 使对所有 $t \in (0, \infty)$, 都有 $k_1 \leq k(t) \leq k_2$.

(A₂) 存在常数 $B \geq A$, 使对所有 $x \in \mathbf{R}$, 都有 $A \leq \varphi(x) \leq B$.

引理 1 设在某区间 $[\tau(a), \sigma(b)]$ 上, 有 $p(t) \geq 0$, $q(t) \geq 0$, $e(t) \leq 0$, 方程(E₁) 的解 $x(t)$ 满足 $x(t) > 0$, 则有:

$$\frac{x(\tau(t))}{x(t)} \geq \left(\frac{\tau(t) - \tau(a)}{t - \tau(a)} \right)^{\frac{Bk_2}{Ak_1}}, t \in [a, \sigma(b)]; \quad (1)$$

$$\frac{x(\sigma(t))}{x(t)} \geq \left(\frac{\sigma(b) - \sigma(t)}{\sigma(b) - t} \right)^{\frac{Bk_2}{Ak_1}}, t \in [\tau(a), b]. \quad (2)$$

证明 由方程(E₁) 及所给两个条件, 有 $(k(t)\varphi(x(t))x'(t))' = e(t) - [p(t)x(\tau(t)) + q(t) \times x(\sigma(t))] \leq 0, t \in [\tau(a), \sigma(b)]$. 所以 $(k(t)\varphi(x(t))x'(t))$ 不增. 当 $t \in (\tau(a), \sigma(b)]$ 时, 由中值定理存在实数 $s \in (\tau(a), t)$ 使得 $x(t) - x(\tau(a)) = x'(s)(t - \tau(a))$, 从而有 $x(t) \geq x(t) - x(\tau(a)) = x'(s)(t - \tau(a)) = \frac{k(s)\varphi(x(s))x'(s)}{k(s)\varphi(x(s))}(t - \tau(a)) \geq \frac{k(t)\varphi(x(t))x'(t)}{k(s)\varphi(x(s))}(t - \tau(a))$, 所以 $\frac{x'(t)}{x(t)} \leq \frac{k(s)\varphi(x(s))}{k(t)\varphi(x(t))} \times \frac{1}{t - \tau(a)} \leq \frac{Bk_2}{Ak_1} \frac{1}{t - \tau(a)}$. 当 $t \in [a, \sigma(b)]$ 时, 上式两边从 $\tau(t)$ 到 t 积分可得式(1) 成立. 类似可证式(2) 成立. 证毕.

同样可证引理 2.

引理 2 设在某区间 $[\tau(a), \sigma(b)]$ 上, 有 $p(t) \geq 0$, $q(t) \geq 0$, $e(t) \geq 0$, 方程(E₁) 的解 $x(t)$ 满足 $x(t) < 0$, 则式(1) 与式(2) 成立.

利用上述关于方程解的估计可得下面结果:

定理 1 设在某区间 $[\tau(a), \sigma(b)]$ 上, 有 $p(t) \geq 0$, $q(t) \geq 0$, $e(t) \leq 0$, 如果存在 $H \in D(a, b)$ 使得

$$\int_a^b [H^2(t)p(t) \left(\frac{\tau(t) - \tau(a)}{t - \tau(a)} \right)^{\frac{Bk_2}{Ak_1}} - H'^2(t)k(t)\varphi(x(t))] dt \geq 0, \quad (3)$$

或

$$\int_a^b [H^2(t)q(t) \left(\frac{\sigma(b) - \sigma(t)}{\sigma(b) - t} \right)^{\frac{Bk_2}{Ak_1}} - H'^2(t)k(t)\varphi(x(t))] dt \geq 0, \quad (4)$$

则对方程(E₁) 的任一解 $x(t)$, 存在 $\xi \in [\tau(a), \sigma(b)]$ 使得 $x(\xi) \leq 0$.

证明 反证法. 不妨设式(3) 成立, 如果存在方程(E₁) 的某个解 $x(t)$ 满足对所有 $t \in [\tau(a), \sigma(b)]$, $x(t) > 0$. 令 $\omega(t) = -\frac{k(t)\varphi(x(t))x'(t)}{x(t)}$, $t \in [a, b]$, 则由方程及引理 1 可知

$$\begin{aligned} \omega'(t) &= \frac{1}{k(t)\varphi(x(t))}\omega^2(t) + p(t)\frac{x(\tau(t))}{x(t)} + q(t)\frac{x(\sigma(t))}{x(t)} - \frac{e(t)}{x(t)} \geq \\ &\frac{1}{k(t)\varphi(x(t))}\omega^2(t) + p(t)\frac{x(\tau(t))}{x(t)} \geq \frac{1}{k(t)\varphi(x(t))}\omega^2(t) + p(t)\left(\frac{\tau(t) - \tau(a)}{t - \tau(a)}\right)^{\frac{Bk_2}{Ak_1}}, t \in [a, b]. \end{aligned}$$

上式两边同乘 $H^2(t)$ 并在 a 到 b 上积分, 注意到 $\int_a^b \omega'(t)H^2(t)dt = \int_a^b H^2(t)d\omega(t) = \omega(t)H^2(t)|_a^b -$

$\int_a^b \omega(t)2H(t)H'(t)dt = -\int_a^b \omega(t)2H(t)H'(t)dt$, 结合式(3) 有

$$\begin{aligned} 0 &\geq \int_a^b \left(\frac{H^2(t)\omega^2(t)}{k(t)\varphi(x(t))} + 2H(t)H'(t)\omega(t) + H^2(t)p(t)\left(\frac{\tau(t) - \tau(a)}{t - \tau(a)}\right)^{\frac{Bk_2}{Ak_1}} \right) dt = \\ &\int_a^b \left(\frac{H(t)\omega(t)}{\sqrt{k(t)\varphi(x(t))}} + \sqrt{k(t)\varphi(x(t))}H'(t) \right)^2 dt + \end{aligned}$$

$$\int_a^b \left(H^2(t) p(t) \left(\frac{\tau(t) - \tau(a)}{t - \tau(a)} \right)^{\frac{Bk_2}{Ak_1}} - k(t) \varphi(x(t)) H'^2(t) \right) dt \geq \int_a^b \left(\frac{H(t) \omega(t)}{\sqrt{k(t) \varphi(x(t))}} + \sqrt{k(t) \varphi(x(t))} H'(t) \right)^2 dt.$$

于是有 $\int_a^b \left(\frac{H(t) \omega(t)}{\sqrt{k(t) \varphi(x(t))}} + \sqrt{k(t) \varphi(x(t))} H'(t) \right)^2 dt = 0$, 从而有 $\omega(t) = -\frac{k(t) \varphi(x(t)) H'(t)}{H(t)}$, $t \in (a, b)$. 再结合 $\omega(t)$ 的定义可得 $\frac{x'(t)}{x(t)} = \frac{H'(t)}{H(t)}$, $t \in (a, b)$, 由此可解得 $x(t) = \frac{x(t_0) H(t)}{H(t_0)}$, 其中 t_0 为 (a, b) 中任一点. 由 $H(a) = H(b) = 0$ 可知 $x(a) = x(b) = 0$, 这与 $x(t) > 0$ 矛盾, 定理证毕.

类似地还有下面结果.

定理 2 设在某区间 $[\tau(a), \sigma(b)]$ 上, 有 $p(t) \geq 0$, $q(t) \geq 0$, $e(t) \leq 0$, 如果存在 $H \in D(a, b)$ 使得式(3)或式(4)成立, 则对方程 (E_1) 的任一解 $x(t)$, 存在 $\eta \in [\tau(a), \sigma(b)]$ 使得 $x(\eta) \geq 0$.

结合上述结论可得下面的振动性结果.

定理 3 设存在区间列 $[\tau(a_n), \sigma(b_n)]$, 使得 $p(t) \geq 0$, $q(t) \geq 0$, $t \in [\tau(a_n), \sigma(b_n)]$; $e(t) \geq 0$, $t \in [\tau(a_{2n}), \sigma(b_{2n})]$; $e(t) \leq 0$, $t \in [\tau(a_{2n+1}), \sigma(b_{2n+1})]$. 如果存在函数列 $H_n \in D(a_n, b_n)$, 使得

$$\int_{a_n}^{b_n} \left(H_n^2(t) p(t) \left(\frac{\tau(t) - \tau(a_n)}{t - \tau(a_n)} \right)^{\frac{Bk_2}{Ak_1}} - k(t) \varphi(x(t)) H_n'^2(t) \right) dt \geq 0, \quad (5)$$

或

$$\int_{a_n}^{b_n} \left(H_n^2(t) q(t) \left(\frac{\sigma(b_n) - \sigma(t)}{\sigma(b_n) - t} \right)^{\frac{Bk_2}{Ak_1}} - k(t) \varphi(x(t)) H_n'^2(t) \right) dt \geq 0, \quad (6)$$

则方程 (E_1) 的所有解振动.

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